Dimension Estimates

5ta ESCUELA DE ANALISIS TOPOLOGICO DE DATOS, November 23

References:

- Learning Algebraic Varieties from Samples by Breiding, Kalisnik, Sturmfels and Weinstein
- Nonlinear Dimensionality Reduction by Lee and Verleysen

1. Consider the set of points $A = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$. What is the Hausdorff dimension of S? What about the box-counting dimension of S?

2. Let $S = \mathbb{Q} \cap [0, 1]$. What is the Hausdorff dimension of S? What about the box-counting dimension of S?

3. Consider a modified Cantor set C' by taking the intersection of sets E'_k with $k \in \mathbb{N}$, where E'_k is defined as follows. Let $a_n = 10^n$ for $n \in \mathbb{N}$, then:

   - $E'_0 = [0, 1]$;
   - obtain E'_k by removing the middle $1/3$ of the intervals in E_{k-1}, whenever $a_{2n} < k \leq a_{2n+1}$ for some $n \in \mathbb{N}$;
   - obtain E'_k by removing the middle $3/5$ of the intervals in E_{k-1}, whenever $a_{2n-1} < k \leq a_{2n}$ for some $n \in N$.

   Effectively this means that we construct E'_k by removing thirds if $1 < k \leq 10$, $100 < k \leq 1000$, $10.000 < k \leq 100.000$, and so on. For other values of k, E'_k is obtained by removing three fifths. C' is then defined as $\cap_{k=0}^{\infty} = E'_k$.
   Show that the upper and the lower box counting dimension for the modified Cantor set do not coincide.