$\Bbbk$ field

①

Prop: $\Bbbk^{r_0} \xrightarrow[r_{01}]{\phi_1} \Bbbk^{r_1} \xrightarrow[r_{12}]{\phi_2} \cdots \xrightarrow[r_{n-1\,n}]{\phi_n} \Bbbk^{r_n}$ (r_{02})

$\Rightarrow \exists$ bases for $\Bbbk^{r_i}$ such that

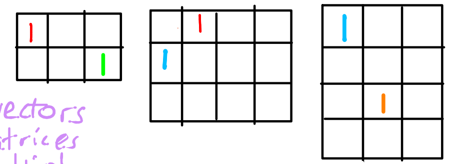
basis vector $\mapsto \begin{cases} \text{basis vector} \\ 0 \end{cases} \mapsto \cdots \mapsto$

$i \qquad i+1 \qquad j$

e.g., including ranks

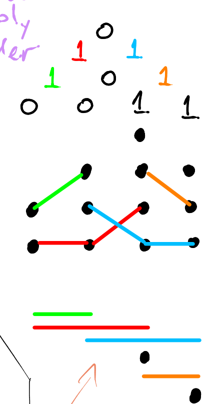
e.g.

$\Bbbk^2 \longrightarrow \Bbbk^3 \longrightarrow \Bbbk^4 \longrightarrow \Bbbk^3$



row vectors so matrices multiply in order

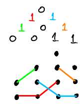
lace array



lacing diagram

use to compute ranks r_{ij}

Def: s_{ij} = # strands starting at i ending at j e.g.



$r_{ij} = \sum_{a \le i \le j \le b} s_{ab}$

e.g. $2 \;\; 1 \;\; 2 \;\; 0 \;\; 1 \;\; 2$

Exercise: express s_{ij} in terms of $\{r_{ab}\}$

(Möbius inversion)

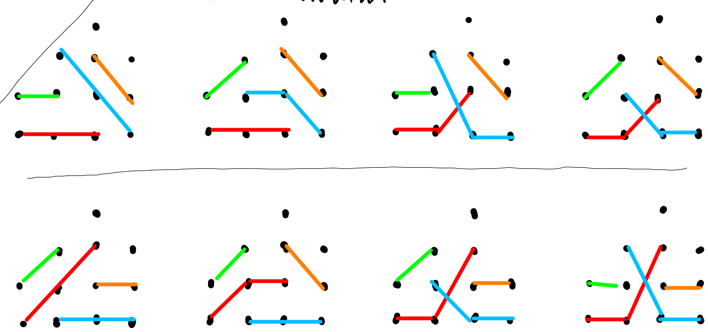
Def [Abeasis – Del Fra 1980] ("diagram")

bar code of $\phi_\cdot$ is multiset of intervals with s_{ij} copies of $[i,j]$ ← **bars**

e.g.

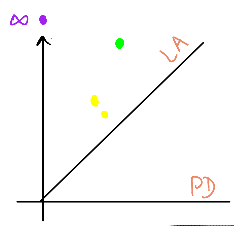
Aside: lacing diagram (lacing diagram)

- realization of bar code on ordered basis
- not unique, even if # crossing is minimal



Thm: bar code is an invariant: every choice of bases yields same bar code.

Def: persistence diagram $\subseteq \mathbb{R}^2$ has one point (α, β) for each bar $\overset{\alpha}{\vdash}\!\!\!-\!\!\!-\!\!\!-\!\!\!\overset{\beta}{\dashv}$



$\Longleftrightarrow$ lace array

Exercise: relation between PD and LA. (rotation, not shear)

Persistent homology

X topological space

$\cup\!|$

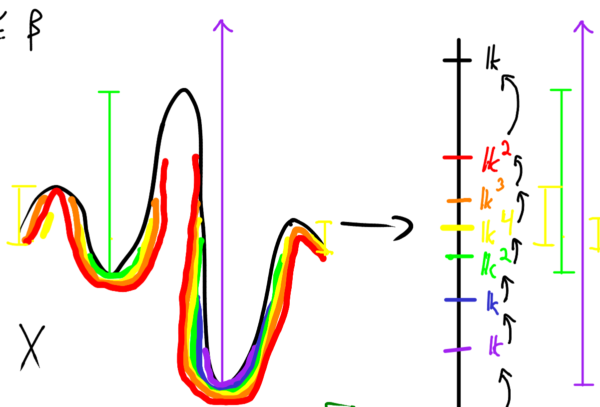
$X_\alpha \quad \alpha \in \mathbb{Z}$, say, or $\alpha \in \mathbb{R} \qquad X_\alpha \subseteq X_\beta$ if $\alpha \le \beta$

e.g. $f : X \to \mathbb{R} \qquad X_\alpha = f^{-1}((-\infty, \alpha])$

set $M_\alpha = \Bbbk\{\text{connected components of } X_\alpha\}$

$= H_0(X_\alpha; \Bbbk)$ persistent homology of filtered space X

$M_\alpha \xrightarrow{\phi_{\alpha\beta}} M_\beta$ if $\alpha \le \beta$ $\qquad H_0 \rightsquigarrow H_i$



topological interpretation of a bar: birth $\rightsquigarrow \overset{\alpha}{\Bbbk} = \cdots = \overset{\beta}{\Bbbk} \leftsquigarrow$ death

2

Def: $\{M_\alpha\}_{\alpha \in \mathbb{R}}$ <u>tame</u> if

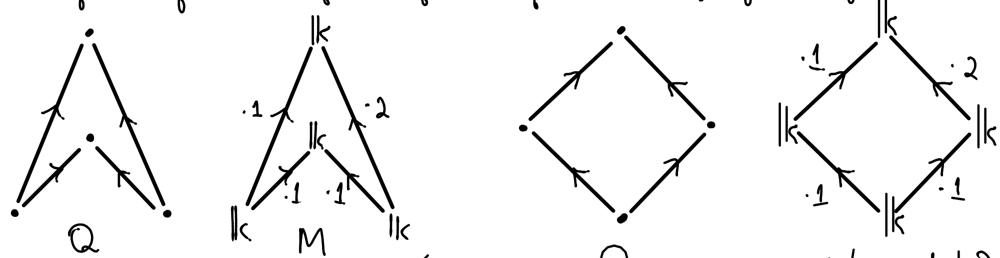
- $\dim_{\Bbbk} M_\alpha < \infty \quad \forall \alpha \in \mathbb{R}$
- $\mathbb{R} = I_1 \cup \cdots \cup I_m$ with $\{M_\alpha\}_{\alpha \in I}$ <u>constant</u>: $M_\alpha \xrightarrow{\sim} M_\beta$ for $\alpha \le \beta$ in I. e.g.

Emphasize: intervals on which M is constant are (very) different than bars.

Def: Fix a poset Q. Then a <u>Q-module</u> is

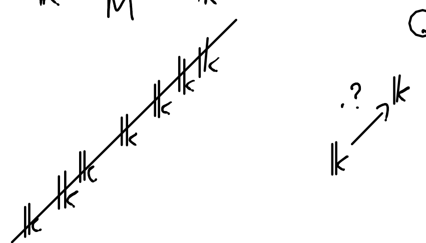
- family $M = \{M_q\}_{q \in Q}$ of vector spaces with
- linear maps $M_q \to M_{q'}$ for $q \preceq q'$
- $M_q \to M_{q''}$ is $M_q \to M_{q'} \to M_{q''}$ for any $q \preceq q' \preceq q''$ no monodromy

(1) e.g.



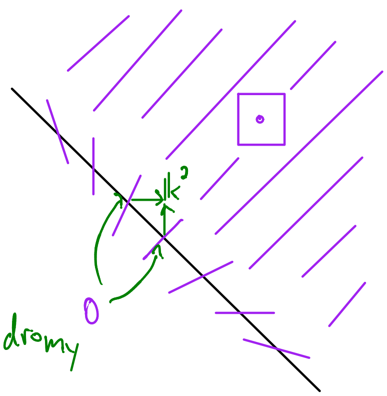
poset module? No: quiver rep...

(2) e.g. $Q = \mathbb{R}^2$ $M =$



$? = 0$ forced

(3) e.g. $Q = \mathbb{R}^2$ $M =$



e.g. $Q = \mathbb{Z}^n \Leftrightarrow M = \mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-module

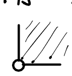
$$\begin{tikzcd} M_{01} \arrow[r, "\cdot x"] & M_{11} \arrow[r, "\cdot x"] & {} \\ M_{00} \arrow[u, "\cdot y"] \arrow[r, "\cdot x"] & M_{10} \arrow[u, "\cdot y"] \arrow[r, "\cdot x"] & {} \end{tikzcd}$$

- standard combinatorial commutative algebra
- finiteness condition: M is noetherian (finitely generated suffices)

$Q = \mathbb{R}^n \Leftrightarrow M = \mathbb{R}^n$-graded $\Bbbk[\mathbb{R}^n_+]$-module

↑ polynomials with real exponents $3x^\pi y^e - 7x^{\sqrt{2}} y$

for statistical purposes — as this morning — wish to do commutative algebra

- not noetherian: maximal ideal $\langle x_i^{a_i} \mid i = 1, \ldots, n$ and $a_i > 0 \rangle$ 

  worse:  $\langle x^a \mid a \ge 0$ and $\sum a_i = 1 \rangle$ uncountably generated
- finiteness hypothesis?

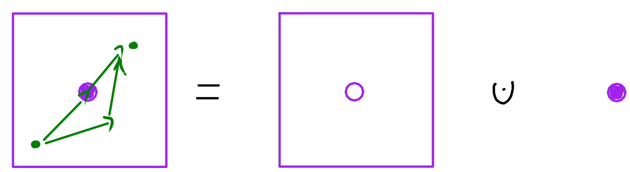
It is not the goal to disallow these objects, but rather to express a finiteness condition that encompasses them: they "should" be allowed.

<u>Def</u>[M]: Q-module M is <u>tame</u> if $\dim_{\Bbbk} M_q < \infty\ \forall q \in Q$ and M admits a finite <u>constant subdivision</u>: partition of Q into

- <u>constant regions</u> A, each with vector space $M_A \xrightarrow{\sim} M_a\ \forall a \in A$ with
- <u>no monodromy</u>: all $a \preceq b$ with $a \in A$ and $b \in B$ induce same $M_A \to M_a \to M_b \to M_B$.

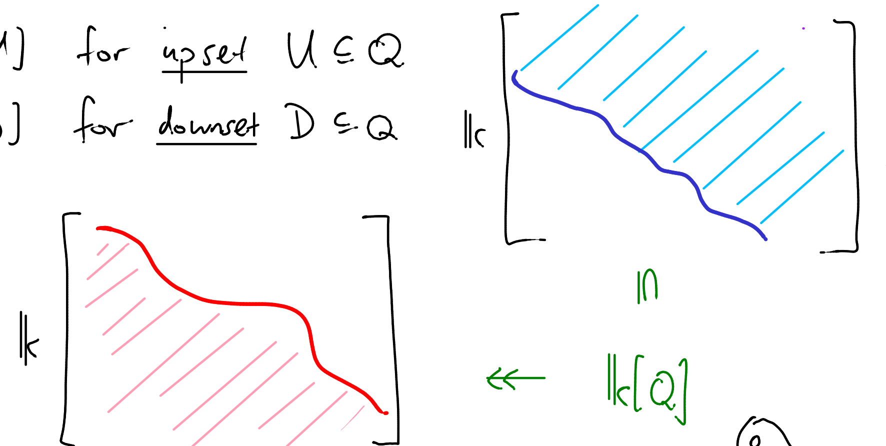
e.g. $Q = \mathbb{R}^2$ $\quad M = \Bbbk_0 \oplus \Bbbk[\mathbb{R}^2]$

constant subdivision

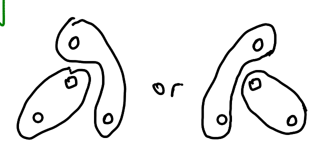


e.g. any Q $\quad M = \Bbbk[U]$ for <u>upset</u> $U \subseteq Q$

or $\Bbbk[D]$ for <u>downset</u> $D \subseteq Q$



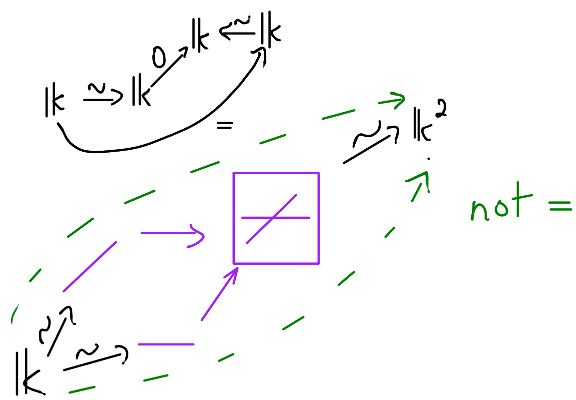
e.g. ① isn't a single constant region, but any partition refining



is a constant subdivision <u>subordinate</u> to M

② isn't tame:

③ isn't tame:

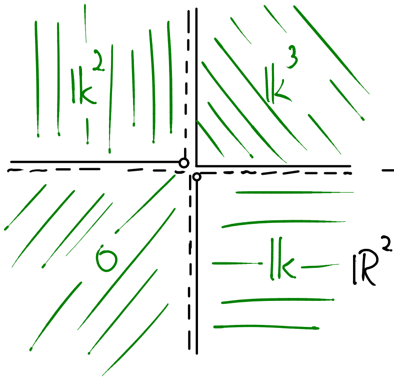


Problem: constant regions need not be partially ordered $\qquad$ e.g. $\Bbbk_0 \oplus \Bbbk[\mathbb{R}^2]$

<u>Def</u>[M]: Q-module M has <u>finite encoding</u> $\pi: Q \to P$ if

- P is finite
- π is a poset morphism
- $M = \pi^* N = \{N_{\pi(q)}\}_{q \in Q}$, the <u>pullback</u> of N along π

e.g.



$\longrightarrow$

$$\begin{tikzcd} \Bbbk^2 \arrow[r] & \Bbbk^3 \\ 0 \arrow[u] \arrow[r] & \Bbbk \arrow[u] \end{tikzcd}$$

P

poset morphism

Thm [M] tame $\Leftrightarrow$ finitely encoded

Pf sketch: $\Leftarrow$: fibers of finite encoding are constant.

$\Rightarrow$: constant region $A \rightsquigarrow$ upset $U(A)$, downset $D(A)$ $\Big\}$ common refinement is partially ordered. $\square$

tame $\Leftrightarrow$ topologically tame;
finitely encoded $\Leftrightarrow$ combinatorially tame;
next lecture: algebraically tame.

Exercise: Does the poset morphism encode $\Bbbk_0 \oplus \Bbbk[\mathbb{R}^2]$?

Yesterday afternoon: topology and combinatorics
today: algebra for computation

e.g. (finite) data structure: (finite) encoding   inefficient: exponential in # params
as long as constant regions are describable: semialgebraic
— beyond scope of this course

Fundamental truth about multiparameter persistence
a.k.a. Q-modules for $Q \neq$ line segment $\Longrightarrow$ no bar code

Fundamental observation about Q-modules: no monodromy axiom $\Rightarrow$ <u>commutative</u> algebra

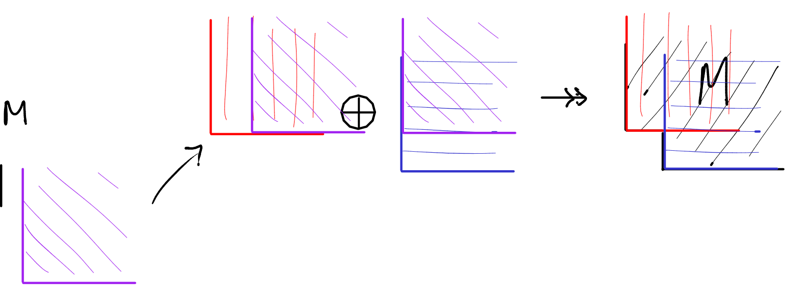
so no complete "combinatorial" invariant, but get

- generators — these turn out to be the hardest concept $/\mathbb{R}$, as on slides
- $\longrightarrow$ presentations / copresentations — e.g. primary decomposition
- resolutions / syzygy theorems — won't say more than is covered on slides

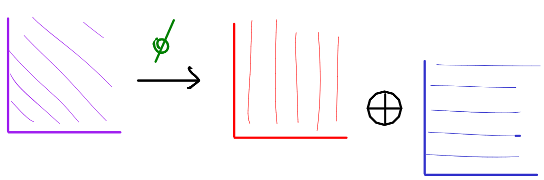
$\boxed{\text{Fix poset } Q \text{ and } Q\text{-module } M.}$

<u>Def</u>: <u>cover</u> of M: surjection $F \twoheadrightarrow M$

<u>presentation</u>: + cover of kernel



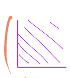
Why?



F_1 $\qquad$ F_0

suffices to recover $M = F_0 / \mathrm{im}(\phi_{F_0 F_1})$

but easy to write down (data structure!):

record
- corner of F_1 ()
- corners of F_0 ( and )
- scalars that say how the corner of F_1 embeds into summands of F_0

$\rightsquigarrow$ <u>monomial matrix</u>
$$\begin{array}{c|cc} \in Q & \begin{bmatrix}0\\1\end{bmatrix} & \begin{bmatrix}1\\0\end{bmatrix} \\ \hline \begin{bmatrix}1\\1\end{bmatrix} & 1 & -1 \end{array}$$
$\in k$

this example is a <u>free</u> presentation

doesn't work as (finite!) data structure for $\mathbb{R}^n$-modules

for <u>upset presentation</u>:
$$\begin{array}{c|ccc} & U_1' & \cdots & U_\ell' \subseteq Q \\ \hline U_1 & \phi_{11} & \cdots & \phi_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ U_k & \phi_{k1} & \cdots & \phi_{k\ell} \end{array}$$
$U_k \subseteq Q$

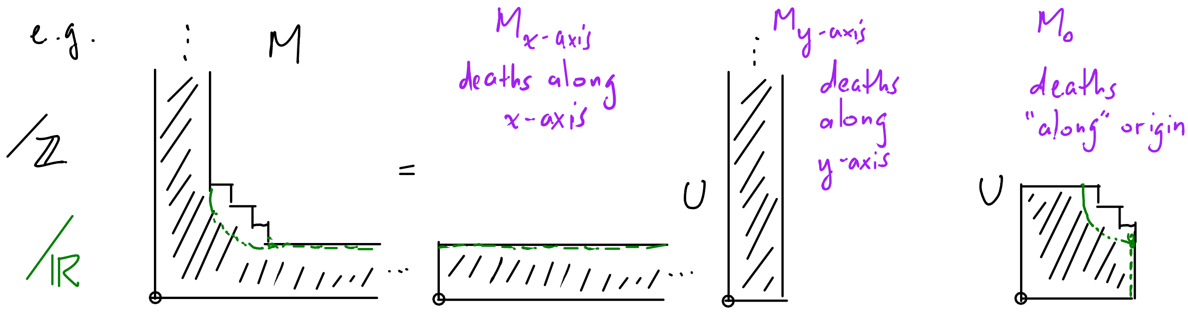
births occur along lower boundaries of these

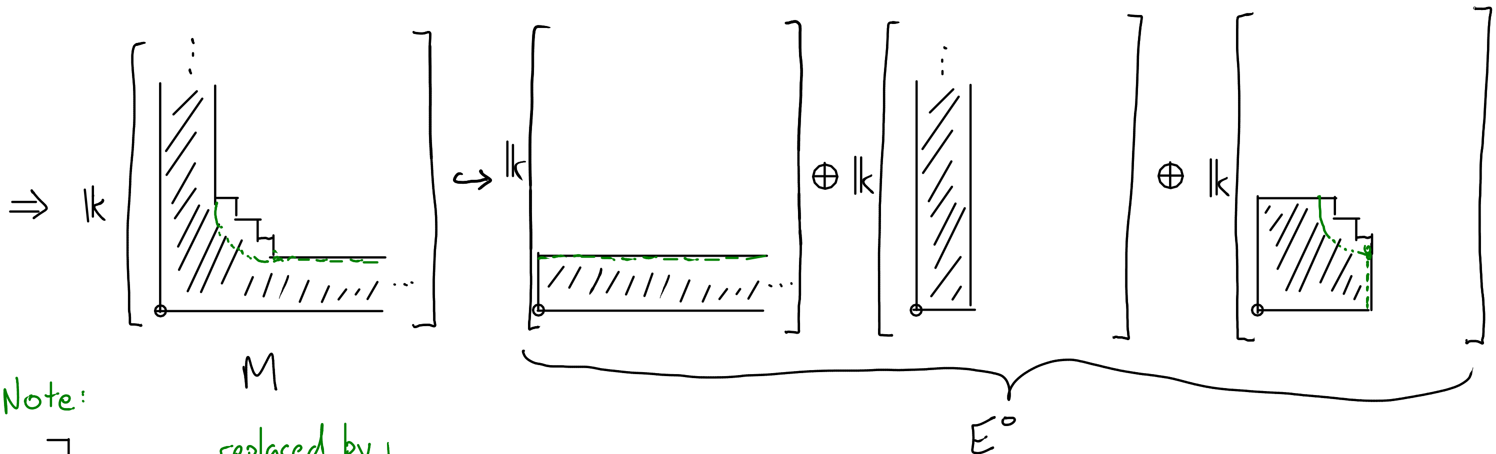
Today: more interested in downset copresentations, or really just hulls.

Def: hull of M: injection $M \hookrightarrow E$ (also called envelope)

copresentation: homomorphism $E^0 \to E^1$ with kernel M

e.g.





Note:

deaths of various types M_x M_y M_0

A rule of life: there are lots of ways to die. Let's make this more precise. (We won't make it completely precise because I won't define socles.)

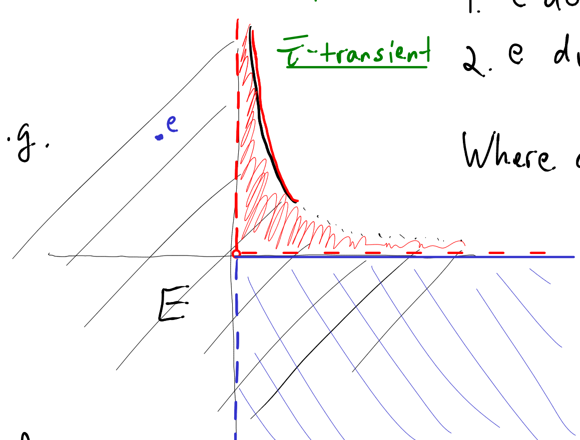
Def: Fix $\tau \subseteq \{1, \ldots, n\}$ and $Q = \mathbb{R}^n$ or $\mathbb{Z}^n$.

Q-module E. An element $e \in E$ is τ-coprimary if

(τ-persistent) 1. e doesn't die when pushed up along τ-axes and

($\bar{\tau}$-transient) 2. e dies when pushed sufficiently along any other axis.

e.g.



Where are the coprimary elements?

$\tau = \emptyset$

$\tau = x\text{-axis}$

$$\begin{array}{ccc} e' & \in & E_{q'} \\ \uparrow & & \uparrow \\ e & \in & E_q \end{array} \Rightarrow e \mid e'$$

Def: E is τ-coprimary if every element in E_q divides a τ-coprimary element $\forall q \in Q$.

Is E coprimary (i.e. τ-coprimary for some τ)?

Yes: $\tau = \emptyset$.

but not τ = x-axis!

Interesting exercises.

1. E coprimary $\Rightarrow$ E is τ-coprimary for unique τ.
2. When $Q = \mathbb{Z}^n$, E coprimary $\Rightarrow$ every element is coprimary.

Thm [M]: $Q = \mathbb{R}^n$ or $\mathbb{Z}^n$.

M finitely encoded $\Rightarrow$ M has a primary decomposition: $M \hookrightarrow \bigoplus_{\tau \subseteq \{1,\dots,n\}} M_\tau$

M is glued together (more or less as a union) from components along coordinate subspaces

with M_τ a τ-coprimary module.

technical term

Note: usual primary decomposition for modules over rings: $M \hookrightarrow \bigoplus_{j=1}^{r} M_j$ where M_j has only one prime associated to it

Hypotheses on Q that suffice for Thm.

- Q = partially ordered abelian group

  $a \le b \Rightarrow a+c \le b+c$

- $Q_+ = \{q \in Q \mid q \ge 0\}$ positive cone

  Lemma: partial ordering $\Updownarrow$ Q_+ pointed submonoid

  polyhedral: # faces $< \infty$

  face = submonoid F with $Q_+ \setminus F$ an ideal.

Same setup, roughly, as

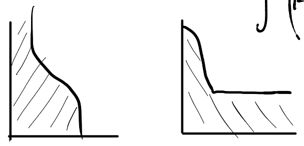
- toric geometry or better
- IH^* on fans [Karu] positivity of cd-index for polytopes: $/\mathbb{R}$ requires real setup.

Why is primary decomposition useful?

1. parameters don't all mean the same thing; differentiate persistence behaviors
2. distances between modules

$$d(M,N) = \sum_\tau w_\tau\, d(M_\tau, N_\tau)$$ weighted sum over faces as indicated by the application

CAN OMIT — useful because, e.g. $\int (\mathrm{Hilb}(M_0) - \mathrm{Hilb}(N_0))$ has better convergence than $\int (\mathrm{Hilb}(M) - \mathrm{Hilb}(N))$



M N

Def [M]: fringe presentation: composite of cover and hull

$$F \twoheadrightarrow M \hookrightarrow E, \quad \text{so} \quad F \xrightarrow{\phi} E \text{ has image } M.$$

birth data (F) death data (E)

as with any (co)presentation, the homomorphism determines M $(= \mathrm{im}\,\phi)$

Particularly informative if

- E carries primary decomp. and F carries $\underbrace{\text{secondary}}_{\text{(Matlis) dual information}}$ decomp

or

- $E = \oplus$ downset modules and $F = \oplus$ upset modules (indicator fringe)

and especially so if either is minimal — notion exists for $\mathbb{R}^n$ or $\mathbb{Z}^n$ but relies on socles and would take one more lecture (+ somewhat more sophisticated algebra)

Summary Thm: $\overset{\text{topological}}{\text{tame}} \Leftrightarrow \overset{\text{combinatorial}}{\text{finitely encoded}}$

$\Leftrightarrow$ finite indicator fringe presentation
$\Leftrightarrow$ finite upset presentation
$\Leftrightarrow$ finite downset copresentation
} algebraic

$\Rightarrow$ primary decomposition
geometric