# Resolving persistent homology

## Lecture 1: Overview

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and Department of Statistical Science

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5ta Escuela de Análisis Topológico de Datos

CIMAT, Guanajuato, México

19 November 2018

# Outline

1. Data
2. Persistent homology
3. Fruit fly wing veins
4. Probability distributions
5. History of persistent homology
6. Biological movitation
7. Topological tameness
8. Presenting poset modules
9. Syzygy theorem
10. Generators and cogenerators
11. Decomposition
12. Future directions

# Collaborators

- Biologists


David Houle (Florida State)

- Mathematicians


Justin Curry (Albany)


Greg Malen (postdoc, Duke)

- Statisticians


Sayan Mukherjee (Duke)


Andrew Nobel (UNC-Chapel Hill)

- Students


Ashleigh Thomas (grad, Duke)


Surabhi Beriwal (undergrad, Duke)

# What kinds of data?

## Shapes

- 1D: curves (in $\mathbb{R}^2$ or $\mathbb{R}^3$, say)
- 2D: photographs
- 3D: MRI, DTI, SPECT, PET, CAT, integrated photo
  - cricket sclerites
  - brain arteries
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- (2+1)D: video (.mp4, .mov, ...)
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- regulatory (genetic)
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- physical: road maps, plant roots, neuronal (dendritic), . . .

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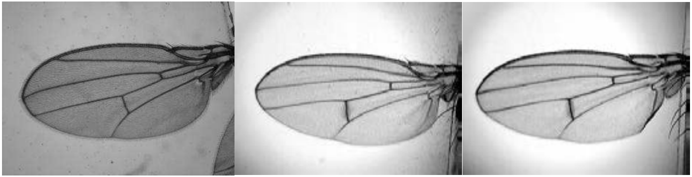
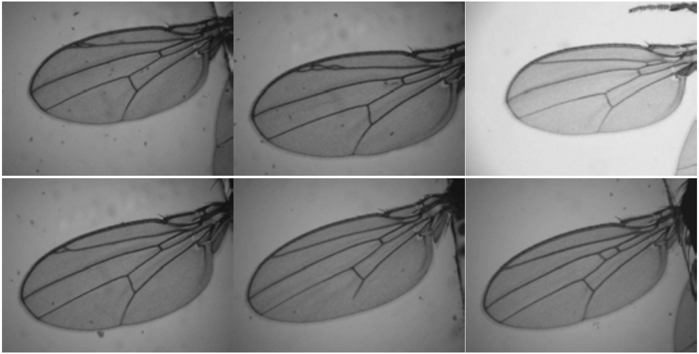
# Fruit fly wings

Normal fly wings [images from David Houle's lab]:



Topologically abnormal veins:



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# A. apoplanos

courtesy Elen Oneal

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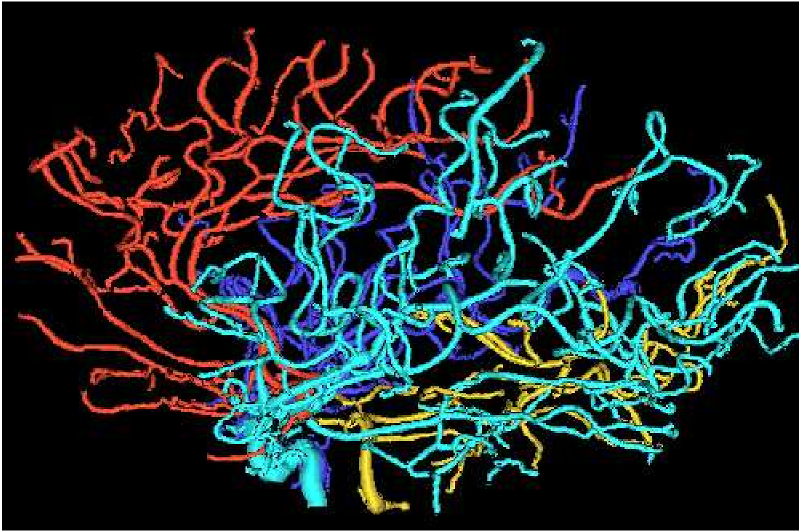

[Bullitt and Aylward, 2002]

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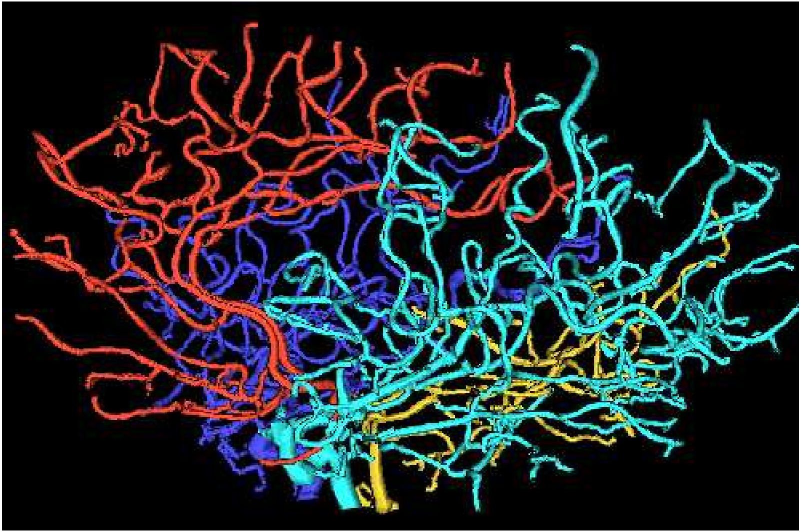

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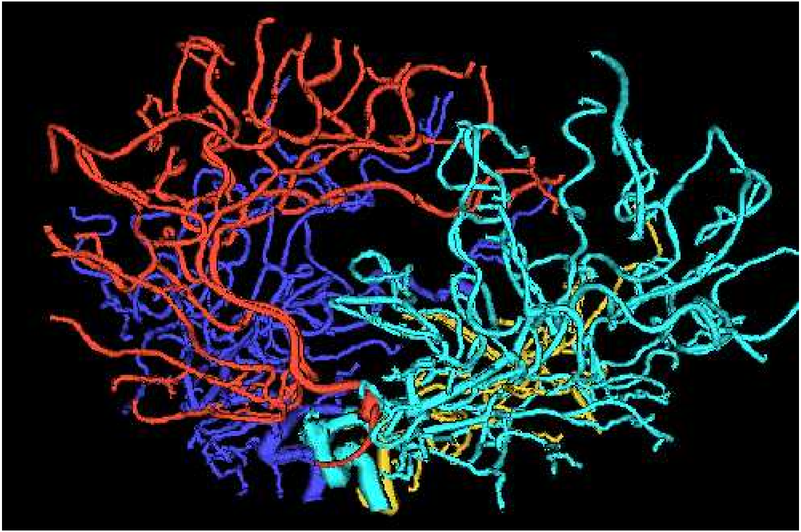

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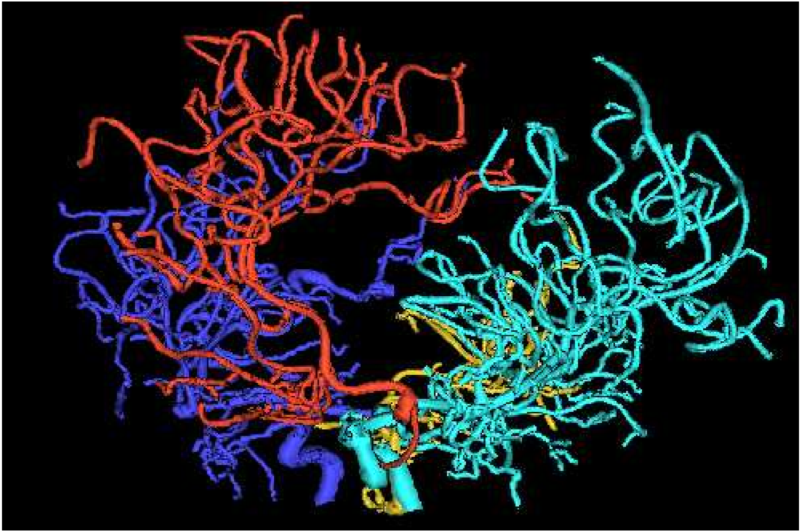

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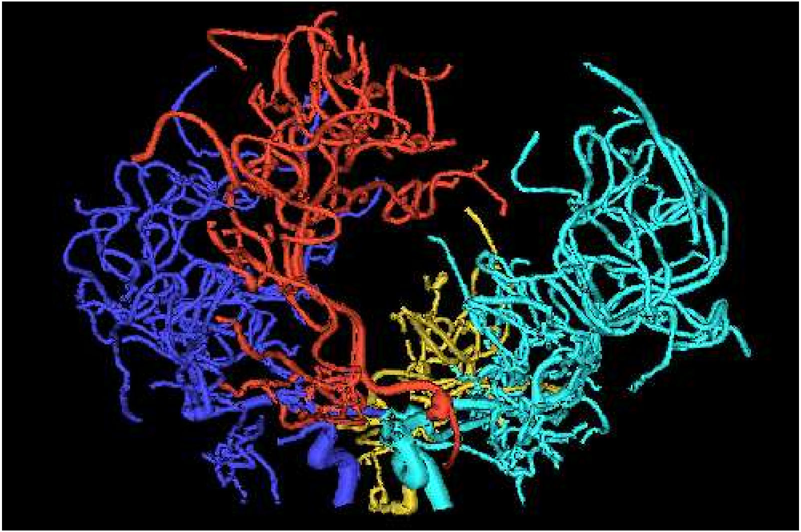

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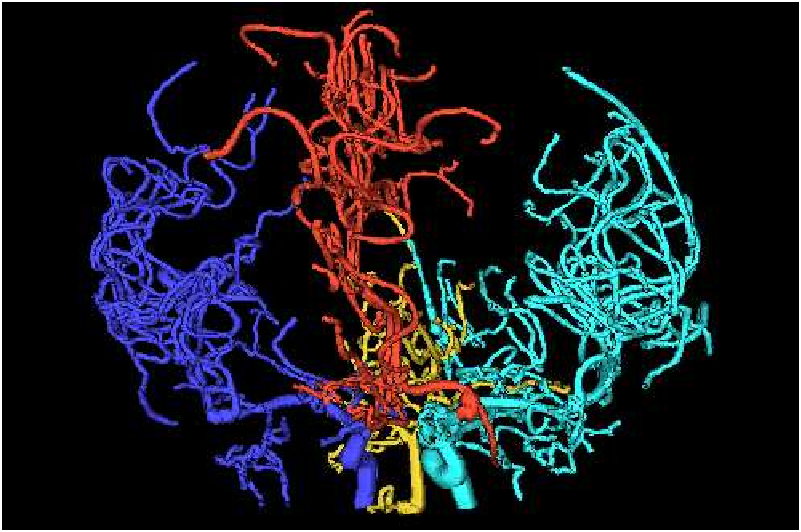

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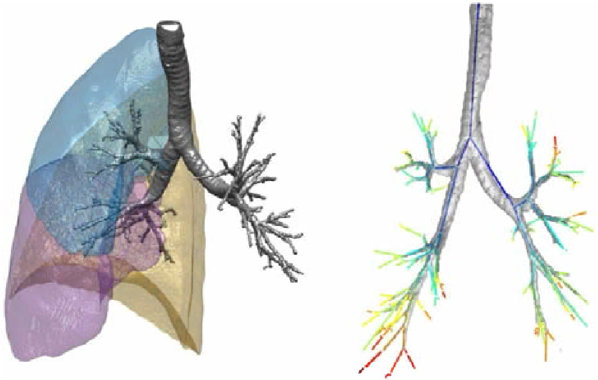
# Lung airways (COPD study)



[Belchi, Pirashvili, Conway, Bennett, Djukanovic, Brodzki 2018]

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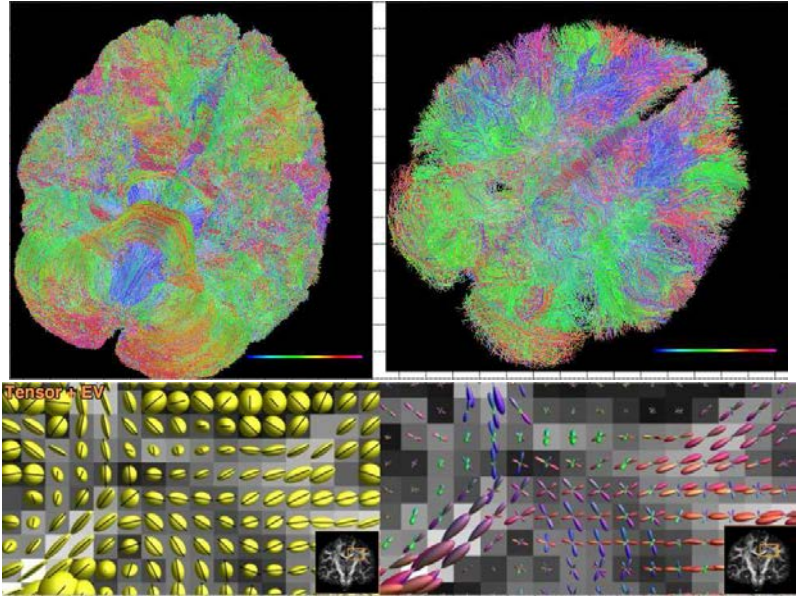
# Streamlines from Diffusion Tensor Imaging




courtesy Zhengwu Zhang

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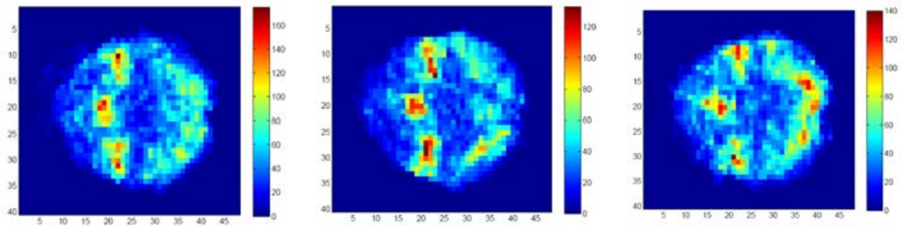

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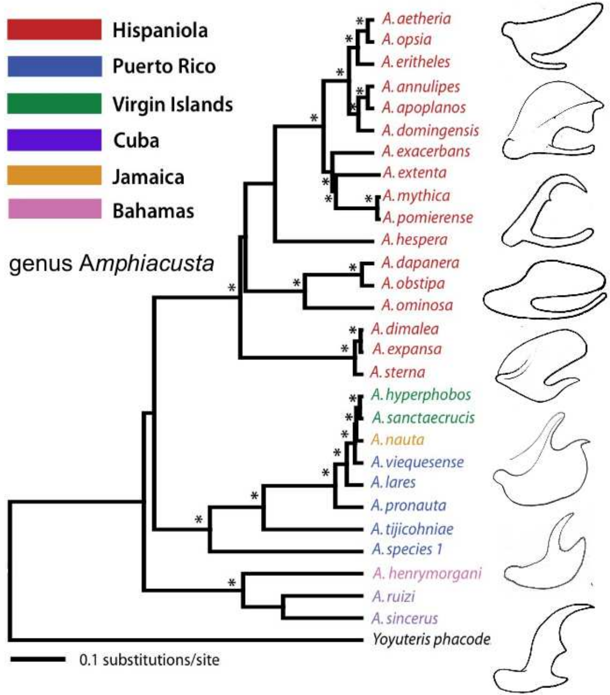


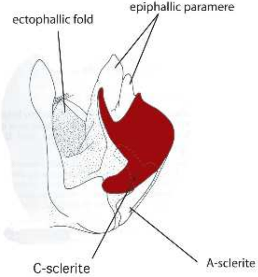


From Oneal, Otte & Knowles, 2010

Drawings by Dan Otte

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# Persistent homology

**Input.** Topological space $X$ filtered by set $Q$ of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

**Def.** $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$.

**Def.** $Q$-module over the poset $Q$:

- family $H = \{H_q\}_{q \in Q}$ of vector spaces over the field $\Bbbk$ with
- homomorphism $H_q \to H_{q'}$ whenever $q \preceq q'$ in $Q$ such that
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**Examples**

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter (“ordinary”) persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter (“ordinary”) persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ — 2 discrete or continuous parameters
- probability distributions: $Q = \mathbb{R}^2$ — 2 continuous parameters
- $Q = \mathbb{Z}^n \Leftrightarrow H = \mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-module
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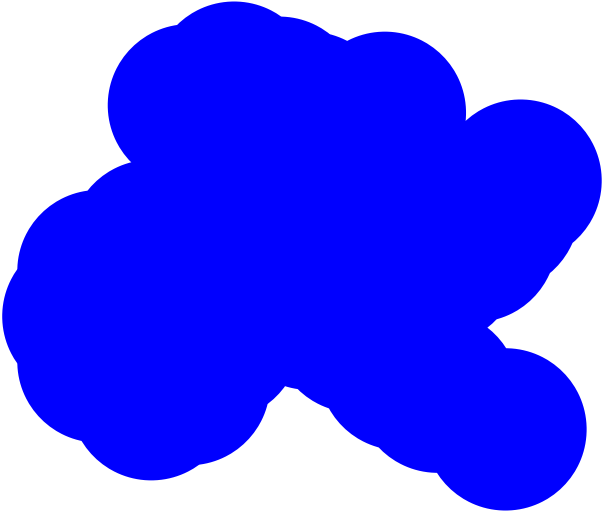

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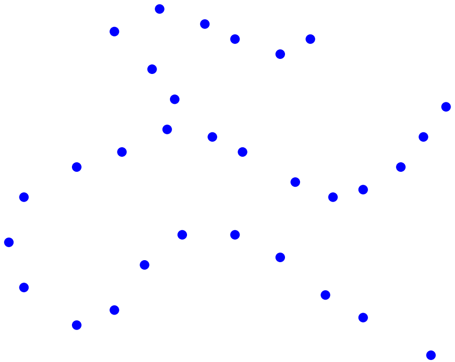

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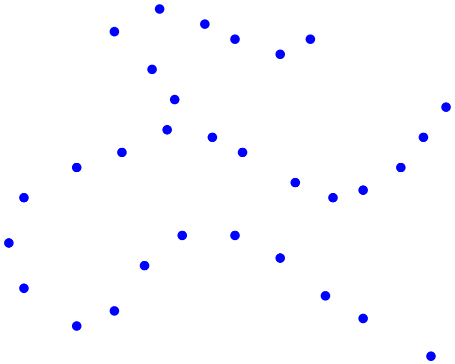

$\dim(H_0) = 31$

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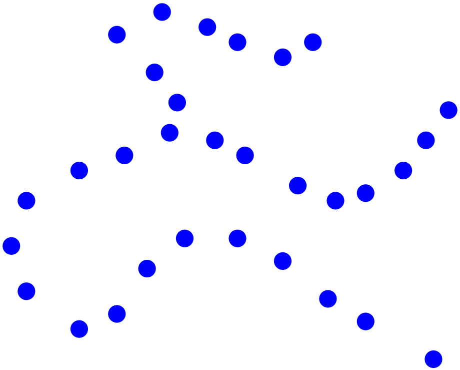

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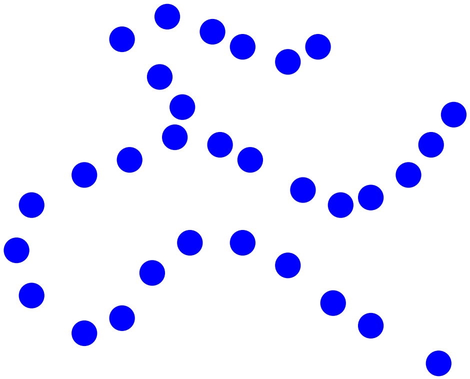

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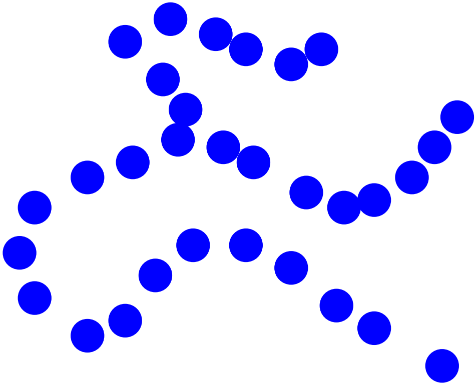

$\dim(H_0) = 26$

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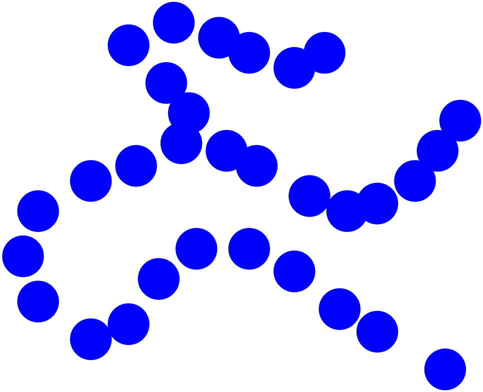

$\dim(H_0) = 21$

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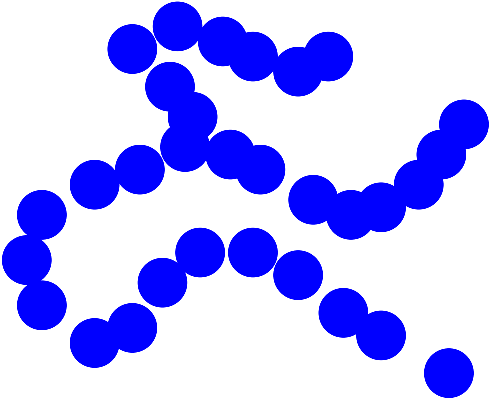

$\dim(H_0) = 12$

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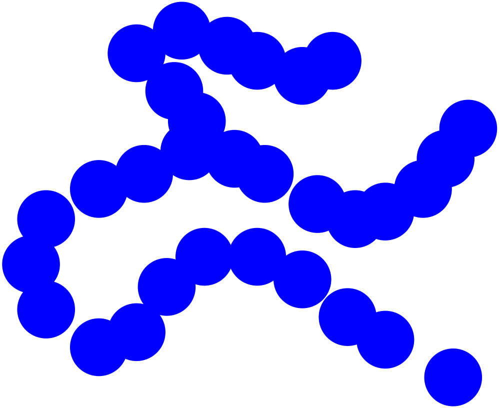

$\dim(H_0) = 6$

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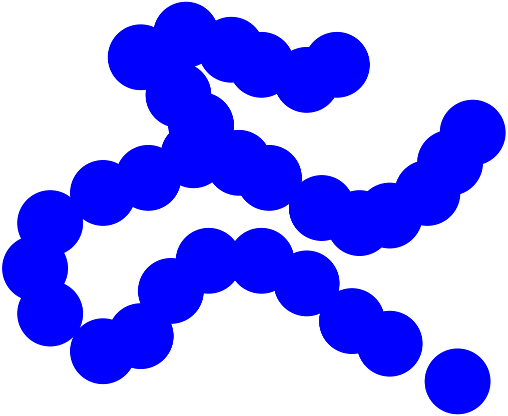

$\dim(H_0) = 2$

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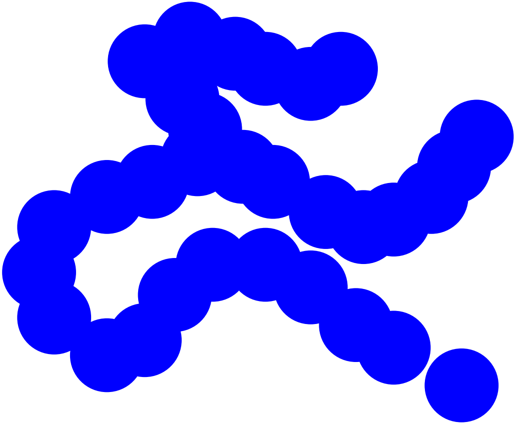

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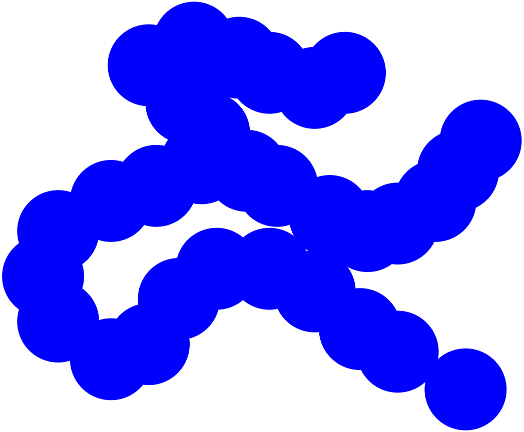

$\dim(H_0) = 1 \qquad \dim(H_1) = 2$

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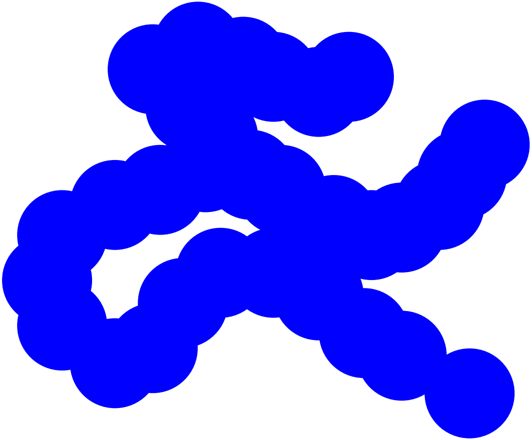

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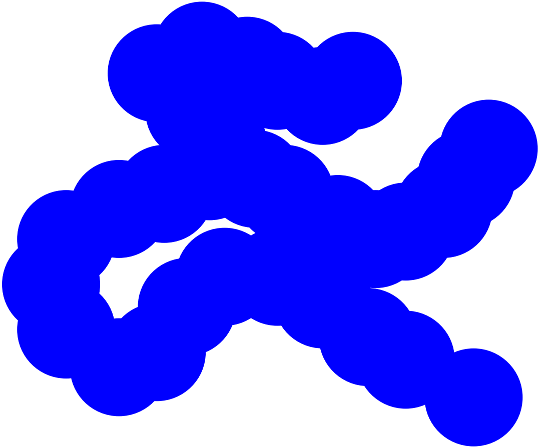

$\dim(H_0) = 1 \qquad \dim(H_1) = 1$

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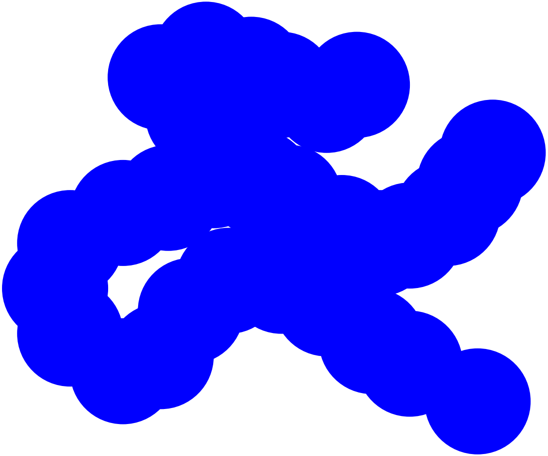

$\dim(H_0) = 1 \qquad \dim(H_1) = 3$

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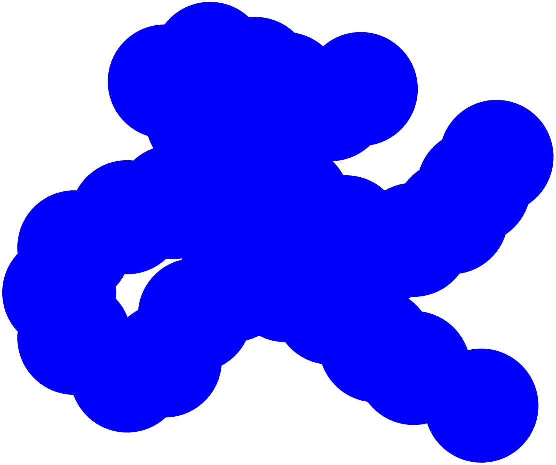

$\dim(H_0) = 1$ $\dim(H_1) = 1$

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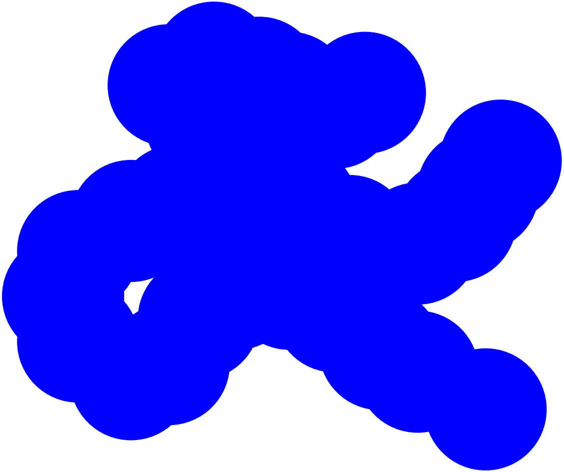

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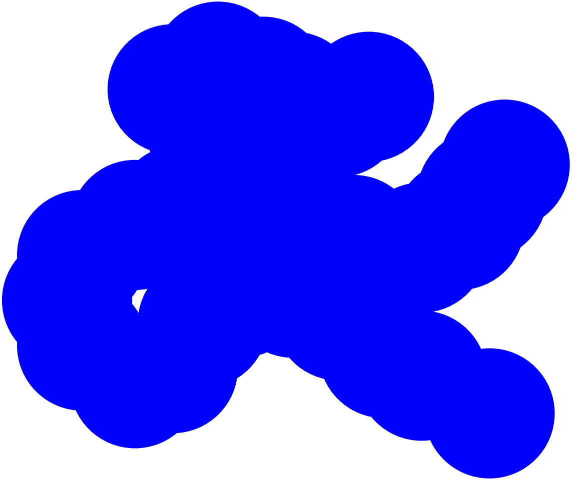

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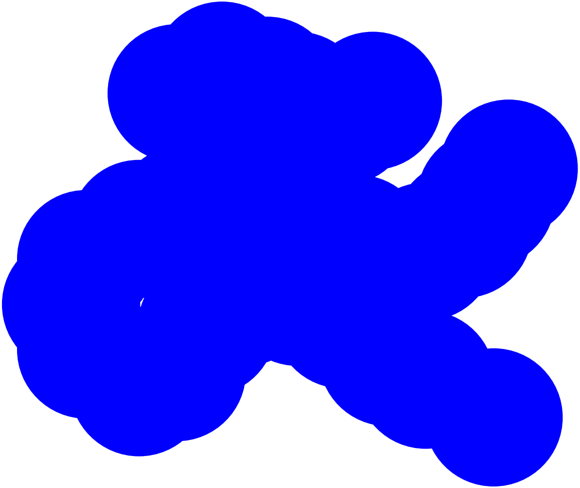

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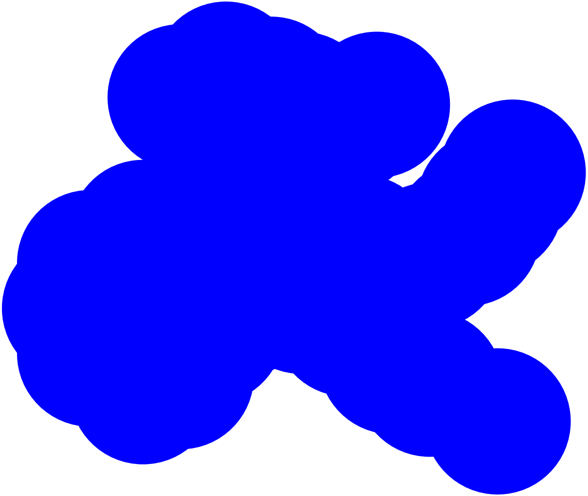

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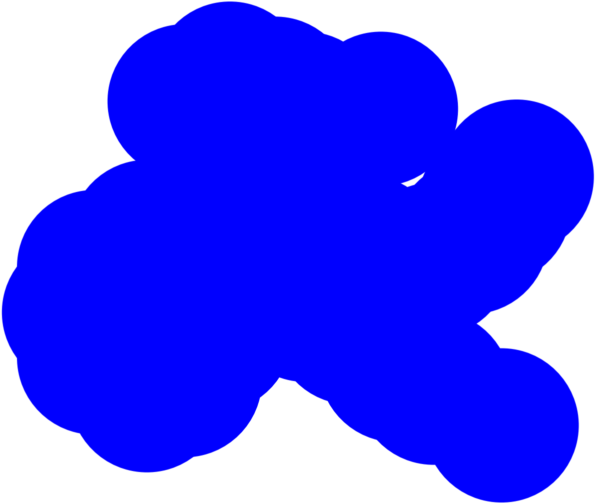

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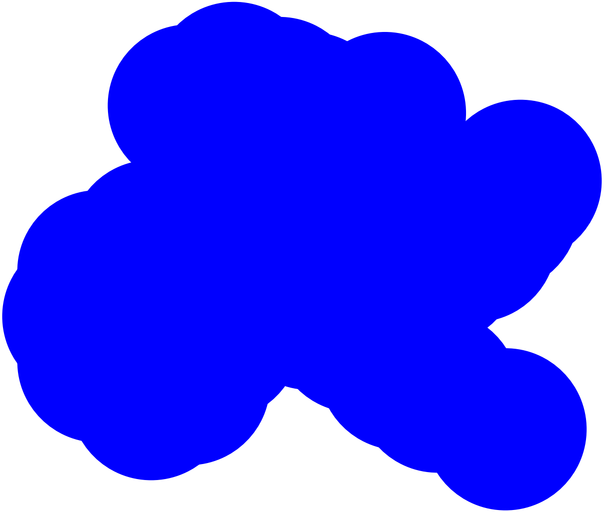

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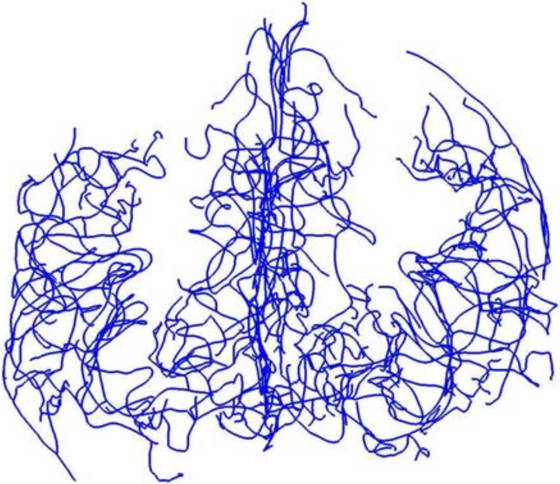

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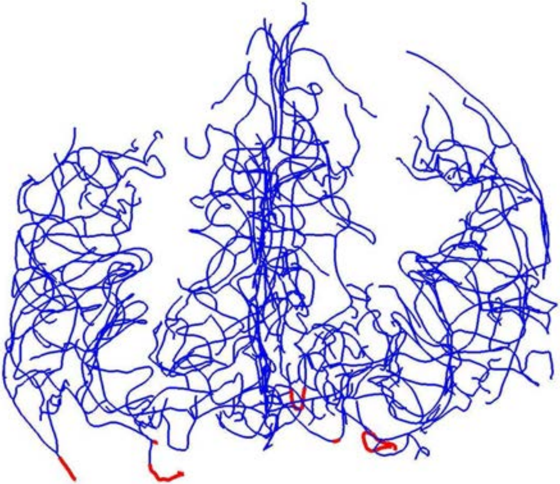

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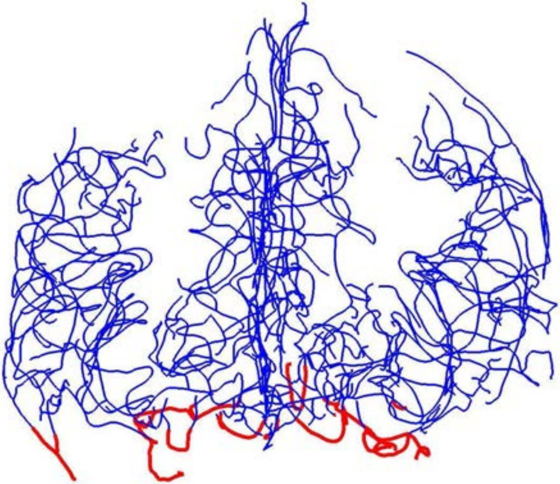

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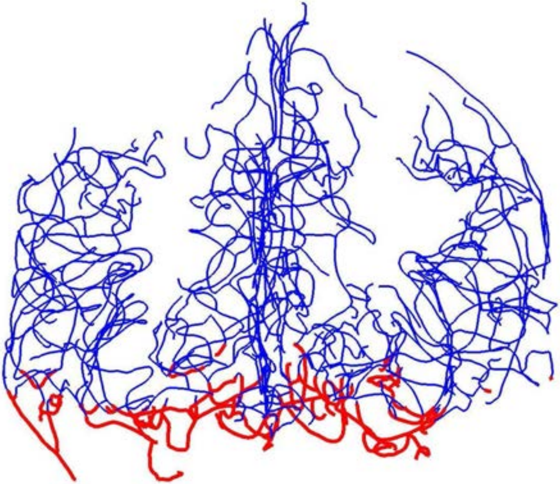

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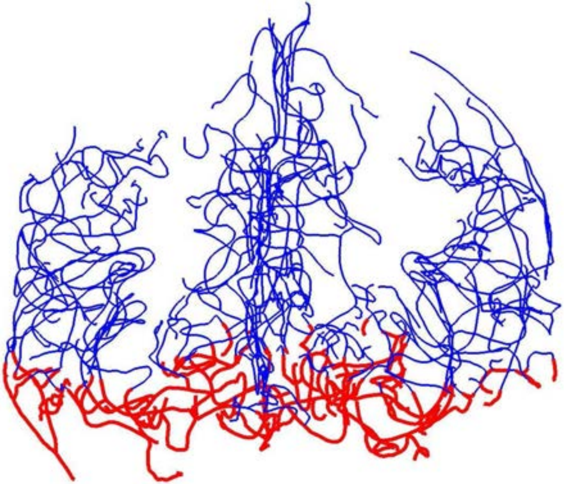

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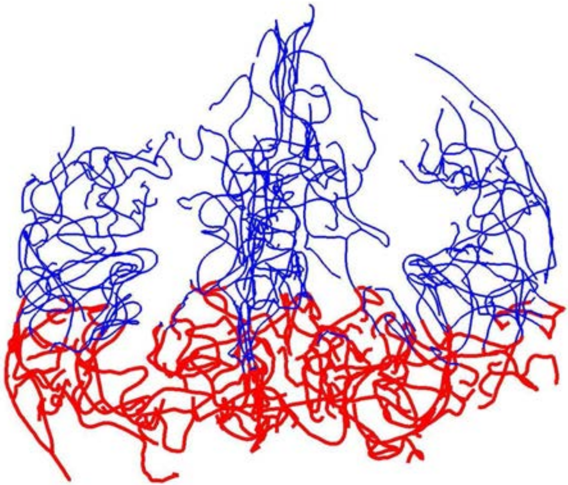

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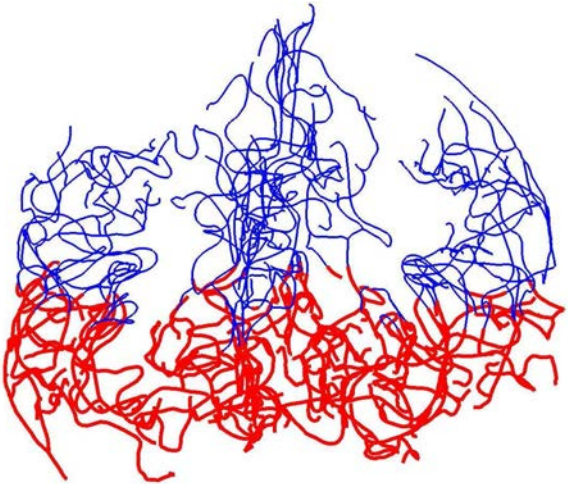

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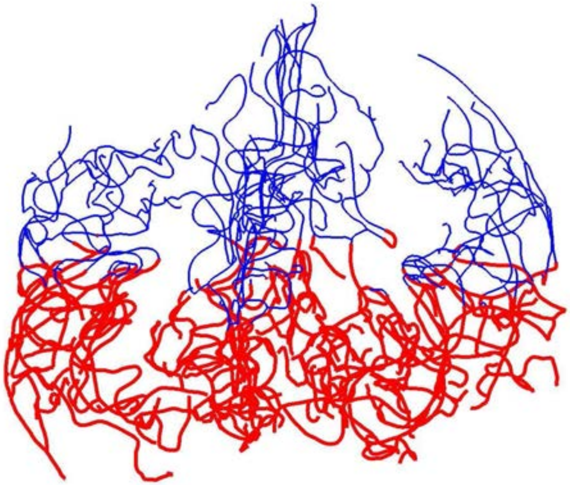

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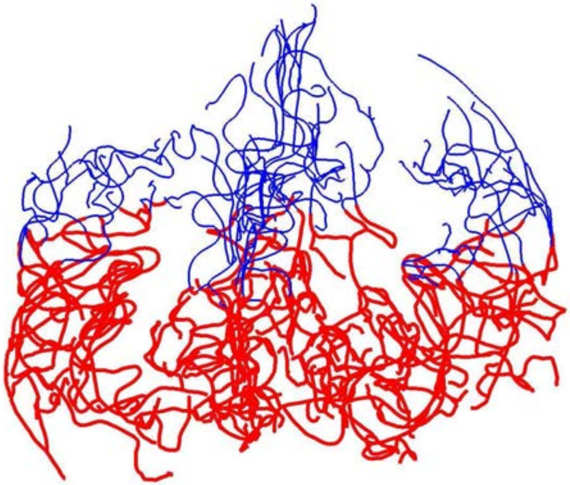

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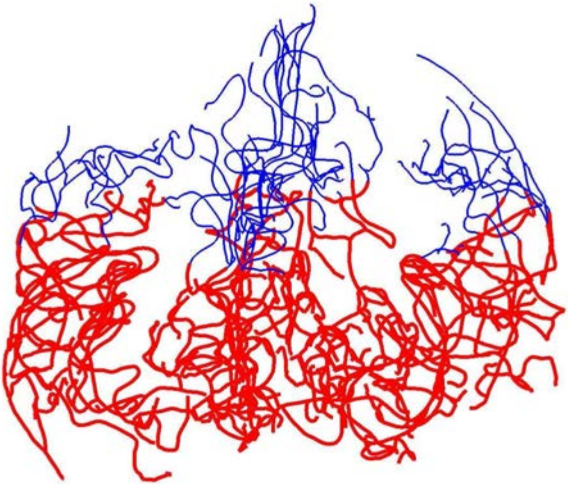

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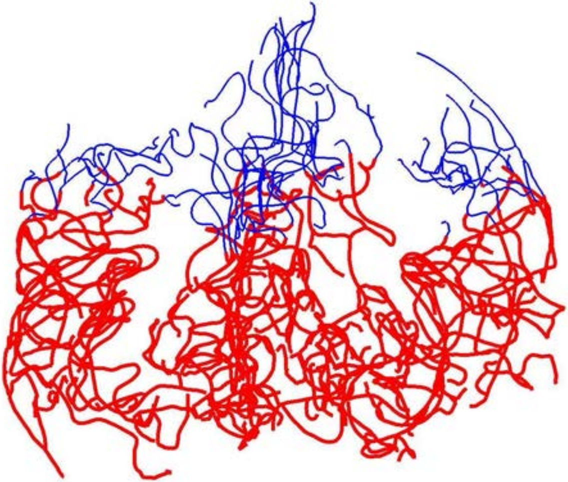

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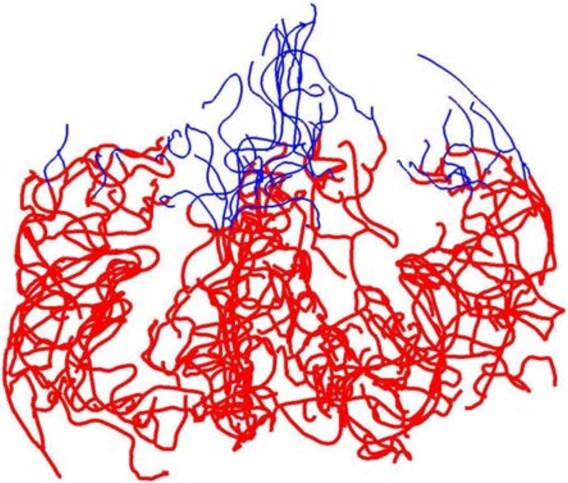

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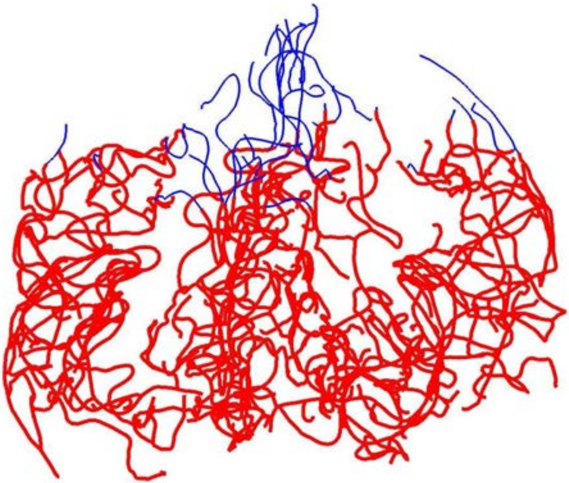

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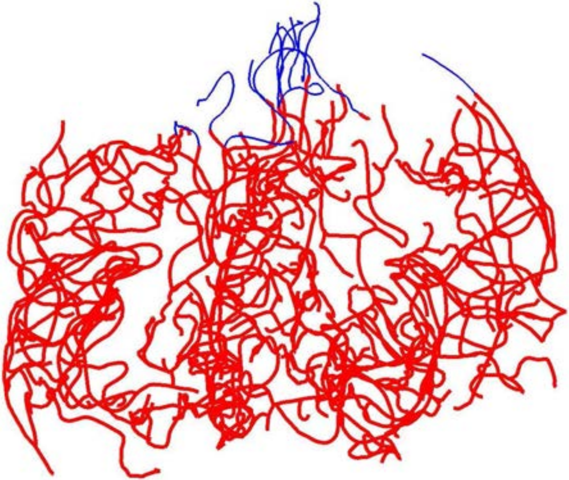

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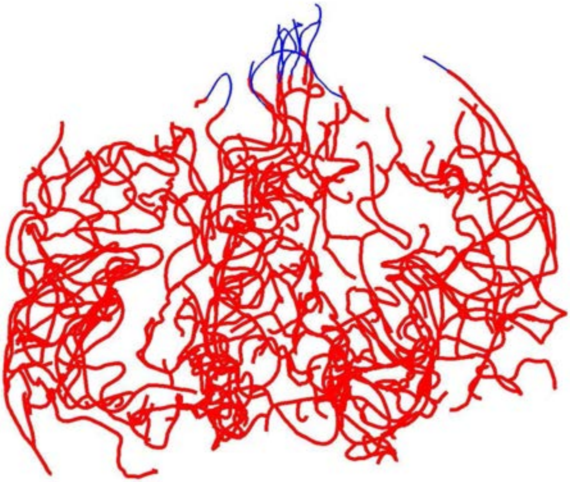

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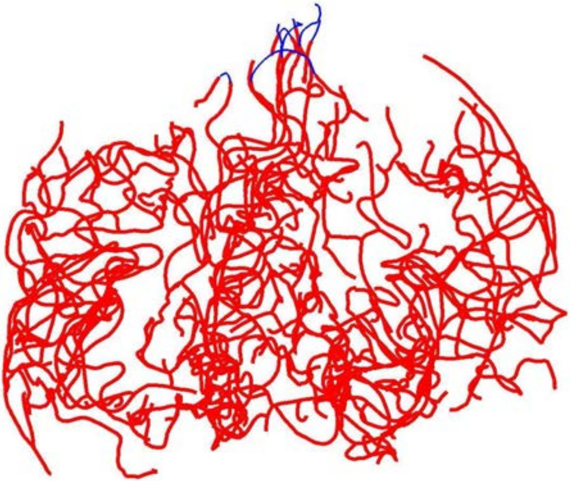

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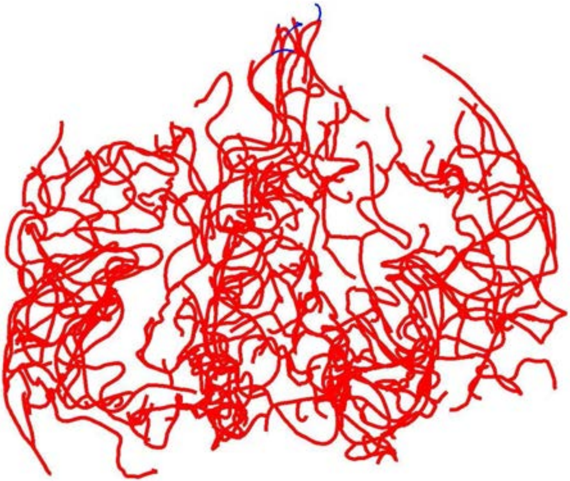

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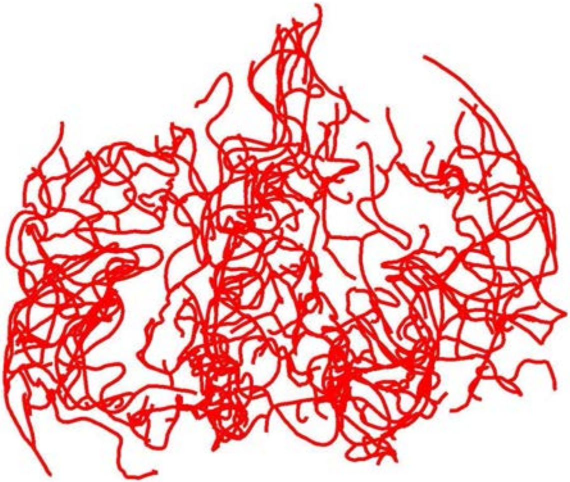

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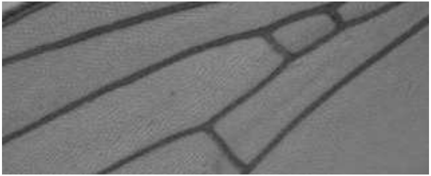
# Example: wing vein persistence [w/Houle, Thomas, Beriwal]

**Example 1.** Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set
- 2nd parameter: distance from edge set



**Sublevel set** $W_{r,s}$ is near edges but far from vertices

**Multiscale summary.** Set $H_{r,s} = H_0(W_{r,s})$ or $H_1(W_{r,s})$

$\mathbb{Z}^2$-module:

$$
\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
\to & H_{r-\varepsilon,s+\delta} & \to & H_{r,s+\delta} & \to & H_{r+\varepsilon,s+\delta} & \to \\
 & \uparrow & & \uparrow & & \uparrow & \\
\to & H_{r-\varepsilon,s} & \to & H_{r,s} & \to & H_{r+\varepsilon,s} & \to \\
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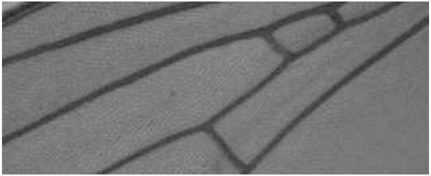

Sublevel set $W_{r,s}$ is near edges but far from vertices

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# Example: wing vein persistence [w/Houle, Thomas, Beriwal]

**Example 1.** Encode fruit fly wing with 2-parameter persistence

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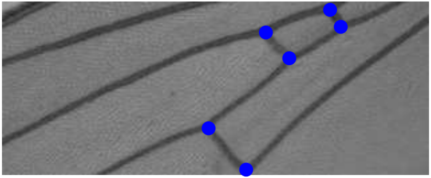

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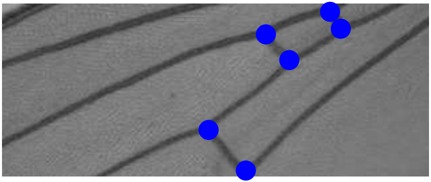

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**Example 1.** Encode fruit fly wing with 2-parameter persistence

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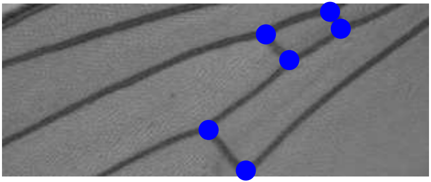

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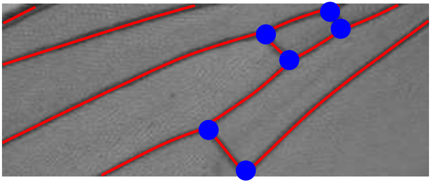

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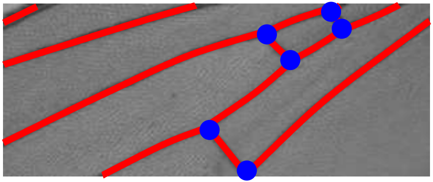

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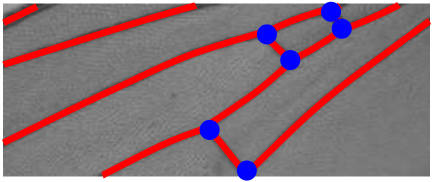

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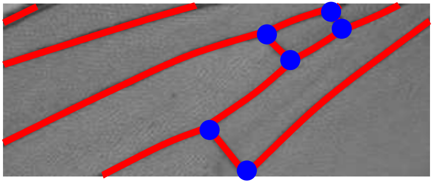

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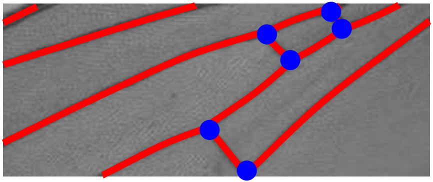

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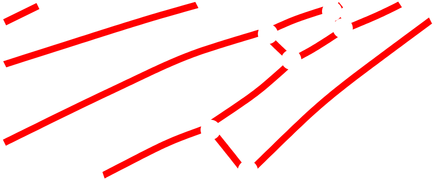

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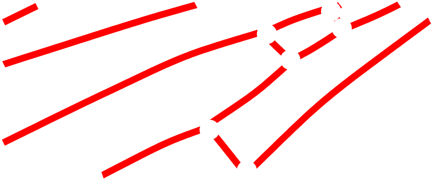

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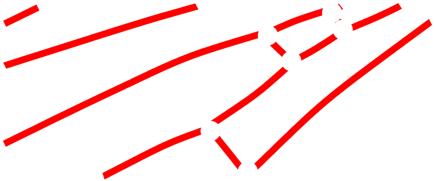

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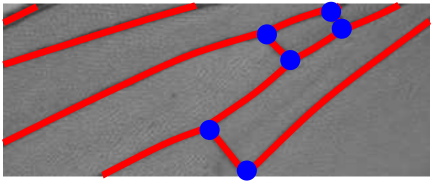

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# Persistent homology

**Input.** Topological space $X$ filtered by set $Q$ of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

**Def.** $\{X_q\}_{q\in Q}$ has persistent homology $\{H_q = H(X_q;\Bbbk)\}_{q\in Q}$. This is a

**Def.** $Q$-module over the poset $Q$:

- family $H = \{H_q\}_{q\in Q}$ of vector spaces over the field $\Bbbk$ with
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## Examples

- points in $\mathbb{R}^n$: $Q = \{0,\ldots,m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0,\ldots,m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ — 2 discrete or continuous parameters
- probability distributions: $Q = \mathbb{R}^2$ — 2 continuous parameters
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# Example: topology of probability distributions

Given probability measure $\mu$ on a space $M$ and kernel function of bandwidth $r$

e.g. • $K_r$ = Gaussian (normal distribution) of variance $r$ on $\mathbb{R}^d$

• $K_r$ = uniform measure on ball of radius $r$ on $\mathbb{R}^d$

Def. Convolution with kernel $K_r$ yields bandwidth $r$ expansion $B_r(\mu) = K_r * \mu$.

Example. • $B_r(\mu_n) \sim B_r(\mu)$ if $\mu_n$ is uniform on an $n$-sample from $\mu$

• $\mu = F(x)dx \Rightarrow B_r(\mu)$ has density $K_r * F(x) = \int_M K_r(y-x)d\mu(y)$

Def. $\nu$ with density function $F$ has support at sensitivity $s$:

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Def. The expansion of $\mu$ to bandwidth $r$ and sensitivity $s$ is $B_r(\mu)_{r^d s} \subseteq M$.

Prop. $\{B_r(\mu)_{r^d s} \mid r \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{R}_{\geq 1}\} \subseteq M$ nested as $r$ and $s$ increase.

Topological Data Analysis (TDA): $B_r(\mu)_{r^d s} \rightsquigarrow$ homology $H_*(B_r(\mu)_{r^d s})$

Def. $\mu$ has $i^{\text{th}}$ bipersistent homology $H_i^{rs}(\mu) = H_i\big(B_r(\mu)_{r^d s}\big)$, an invariant of $\mu$

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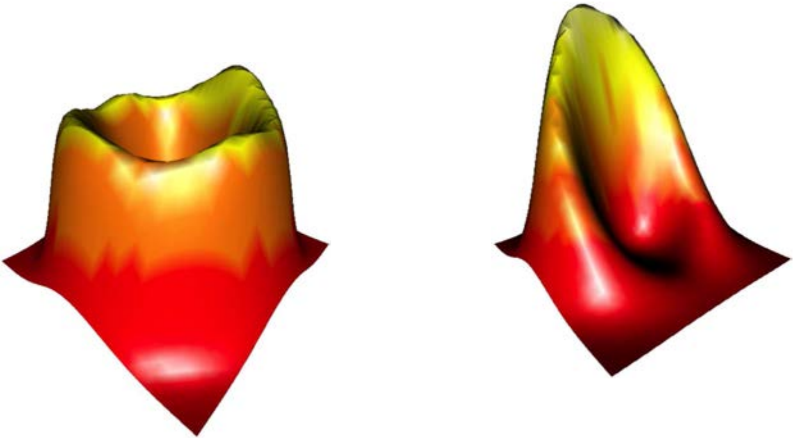
# Topology of probability distributions



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e.g. • $K_r$ = Gaussian (normal distribution) of variance $r$ on $\mathbb{R}^d$

• $K_r$ = uniform measure on ball of radius $r$ on $\mathbb{R}^d$

Def. Convolution with kernel $K_r$ yields bandwidth $r$ expansion $B_r(\mu) = K_r * \mu$.

Example. • $B_r(\mu_n) \sim B_r(\mu)$ if $\mu_n$ is uniform on an $n$-sample from $\mu$

• $\mu = F(x)dx \Rightarrow B_r(\mu)$ has density $K_r * F(x) = \int_M K_r(y - x)d\mu(y)$

Def. $\nu$ with density function $F$ has support at sensitivity $s$:

$$\nu_s = \{x \in M \mid F(x) \geq 1/s\}.$$

Def. The expansion of $\mu$ to bandwidth $r$ and sensitivity $s$ is $B_r(\mu)_{r^d s} \subseteq M$.

Prop. $\{B_r(\mu)_{r^d s} \mid r \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{R}_{\geq 1}\} \subseteq M$ nested as $r$ and $s$ increase.

Topological Data Analysis (TDA): $B_r(\mu)_{r^d s} \rightsquigarrow$ homology $H_*(B_r(\mu)_{r^d s})$

Def. $\mu$ has $i^{\text{th}}$ bipersistent homology $H_i^{rs}(\mu) = H_i\big(B_r(\mu)_{r^d s}\big)$, an invariant of $\mu$

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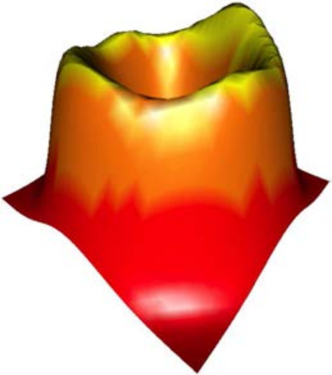

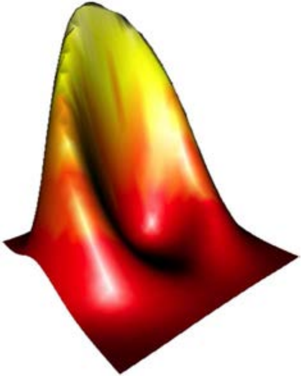

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# Persistent homology

**Input.** Topological space $X$ filtered by set $Q$ of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

**Def.** $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$. This is a

**Def.** $Q$-module over the poset $Q$:

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## Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
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## Ordinary persistence

- traces back to [Morse 1940s]
- formally invented [Frosini, Landi 1999], [Robins 1999]
- efficient computation [Edelsbrunner, Letscher, Zomorodian 2002]
- further theoretical developments [too many to list; mostly 2006– ]
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  - apply standard combinatorial commutative algebra (i.e., $\mathbb{Z}^n$-modules)

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- restrict to parameter lines [Cagliari, Di Fabio, Ferri, Lesnick, Wright,. . . 2010– ]
- finitely presented $\Rightarrow$ Krull–Schmidt: $\bigoplus$ indecomposables [Lesnick 2011]
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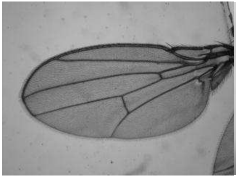
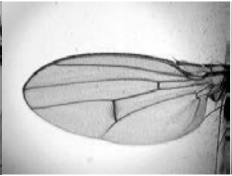
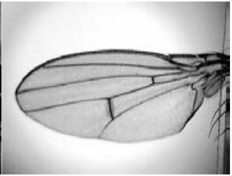

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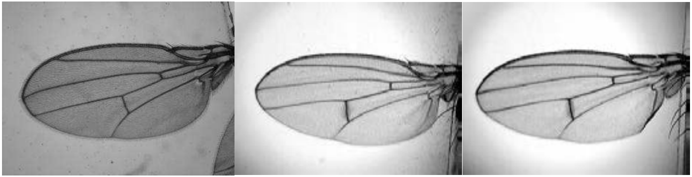
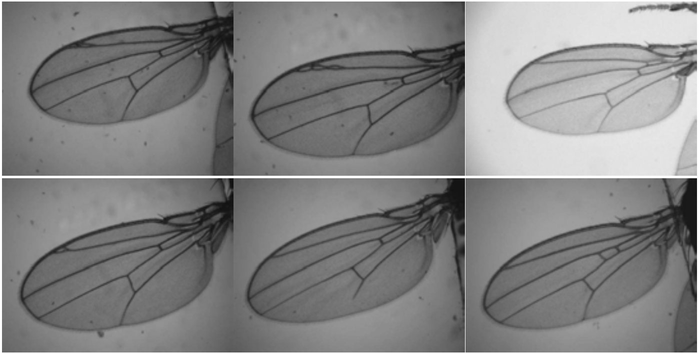


Topologically abnormal veins:



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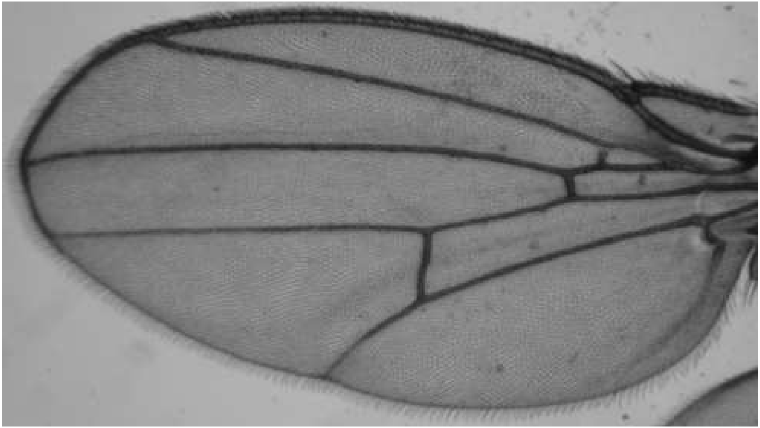
photographic image



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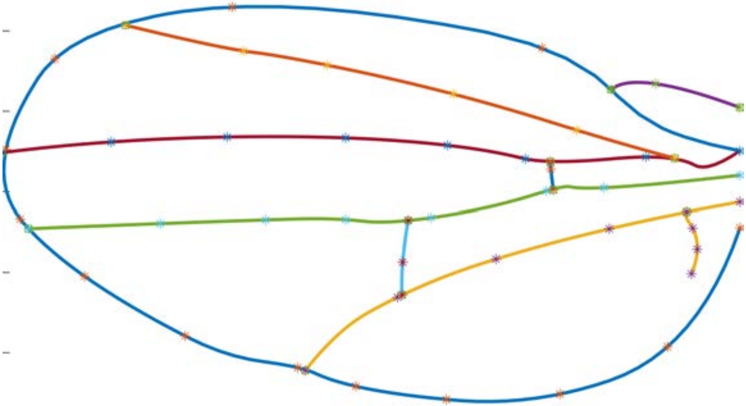
spline



# Biological background

## What generates topological novelty?

[Houle, et al.]: selecting for certain continuous wing vein deformations yields

- skew toward more oddly shaped wings, but also
- much higher rate of topological novelty

**Hypothesis.** Topological novelty arises when directional selection pushes continuous variation in a developmental program beyond a certain threshold.

## Test the hypothesis

- "plot" wings in "form space"
- determine whether topological variants lie "in the direction of" continuous shape selected for, and at the extreme in that direction

**Goal.** Statistical analysis encompassing topological vein variation, giving appropriate weight to new singular points in addition to varying shape

- compare phenotypic distance to genotypic distance; needs
- metric specifying distance between topologically distinct wings

**To proceed.** Statistics with fly wings as data objects $\rightsquigarrow$ statistics with multiparameter persistence diagrams as data objects

**Need.** Data structures, algorithms, theoretical guarantees

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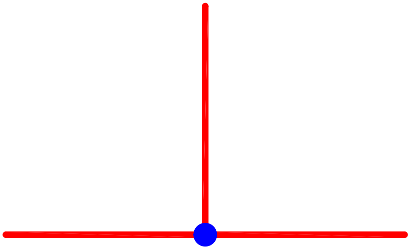

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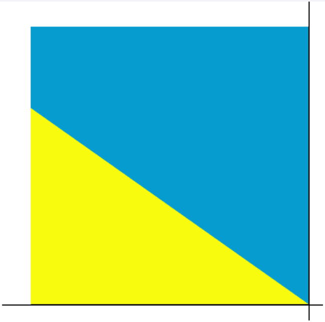

The $(r, s)$-plane $\mathbb{R}^2$

## Observations

- finitely many regions
- stratification alters persistence module
- discrete approximates algebraic

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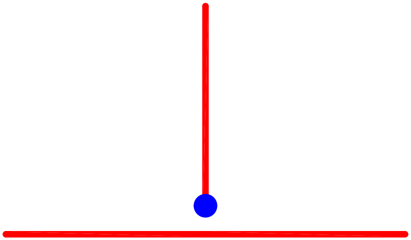

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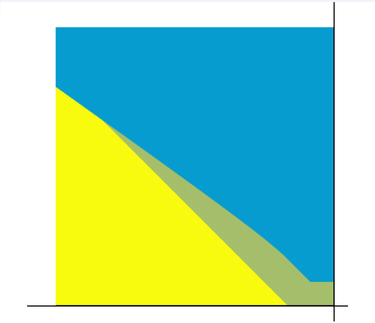

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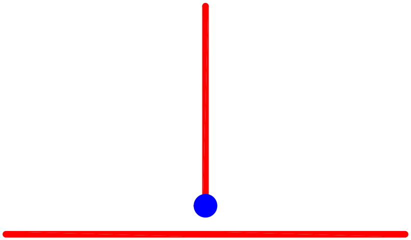

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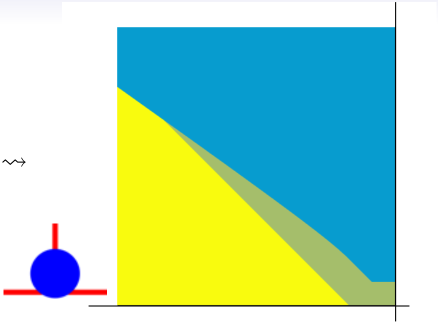

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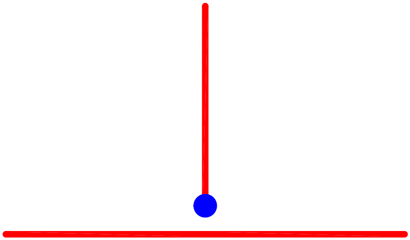

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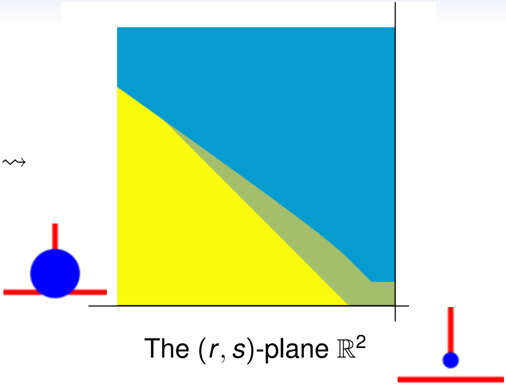

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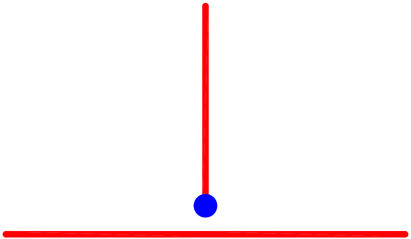

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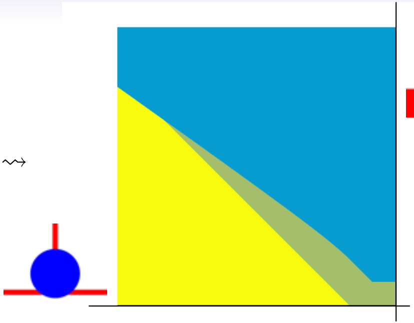

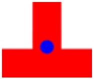

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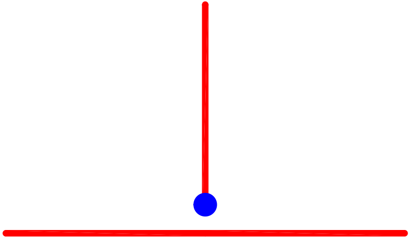

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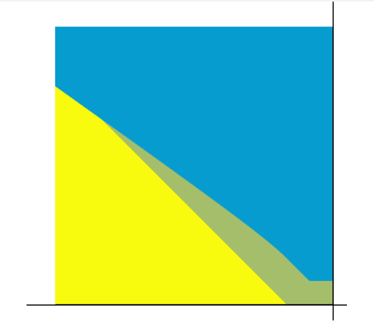

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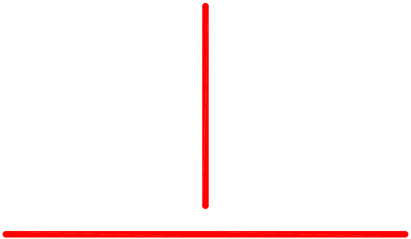

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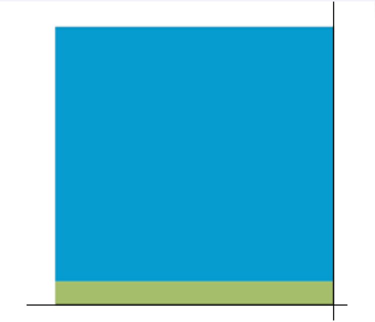

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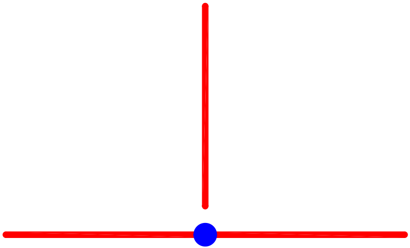

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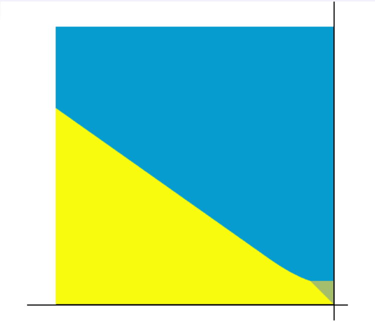

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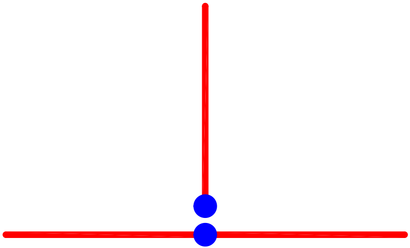

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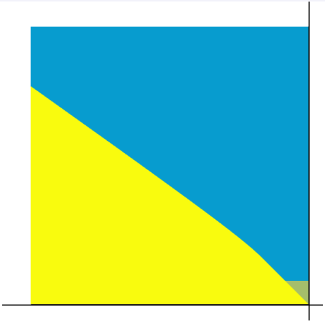

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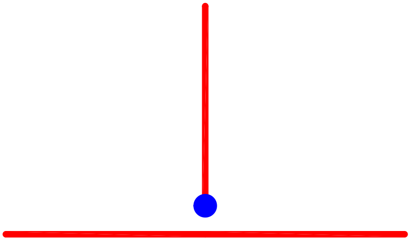

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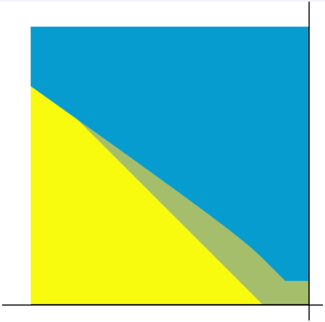

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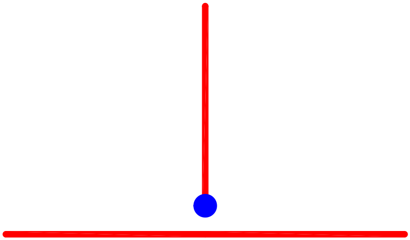

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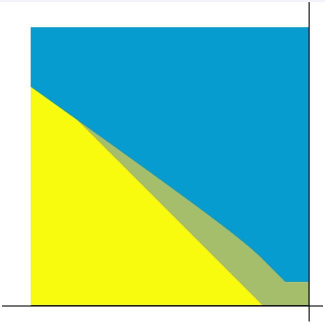

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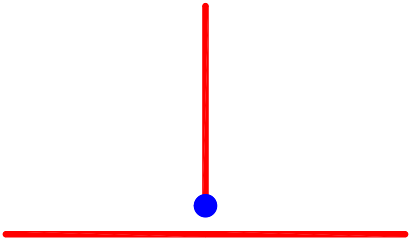

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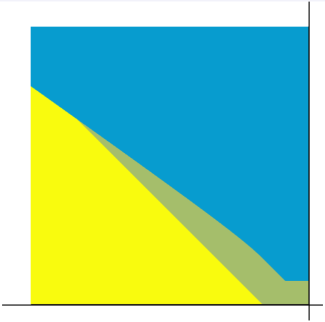

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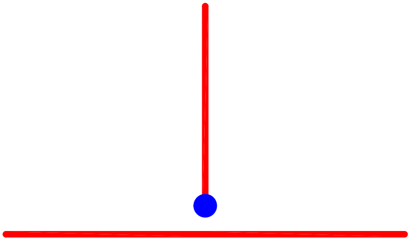

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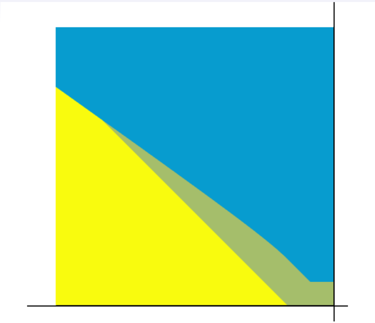

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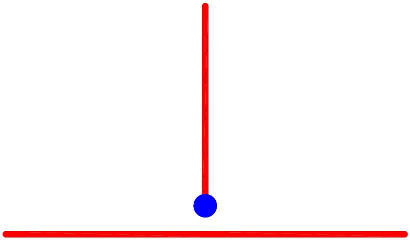

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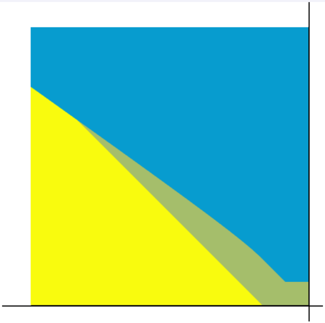

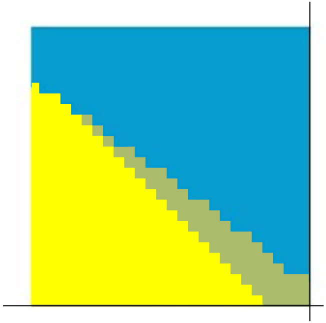

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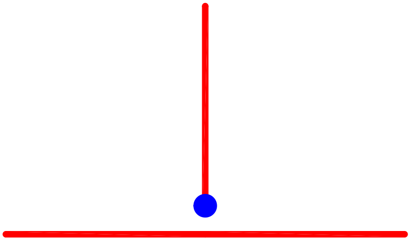

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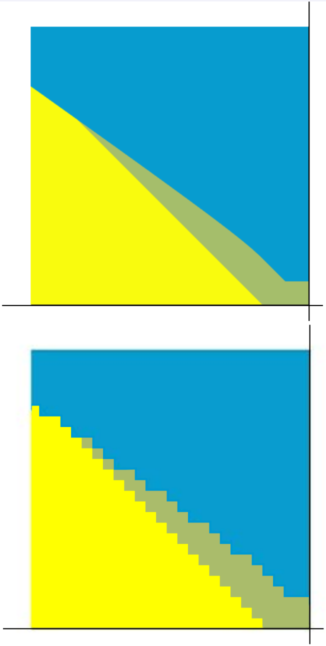

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# Topological tameness

Finiteness conditions:
- $\mathbb{Z}^n$-modules: finitely generated $\Leftrightarrow$ noetherian
- $\mathbb{R}^n$-modules from data analysis $\leftrightarrow$ ??

Def [–]. *H* admits a constant subdivision if *Q* is partitioned into

- constant regions $A \rightsquigarrow$ vector space $H_A \xrightarrow{\sim} H_{\mathbf{a}}$ for all $\mathbf{a} \in A$ with
- no monodromy: all comparable pairs $\mathbf{a} \preceq \mathbf{b}$ with $\mathbf{a} \in A$ and $\mathbf{b} \in B$ induce the same composite $H_A \to H_{\mathbf{a}} \to H_{\mathbf{b}} \to H_B$.

*H* is tame if $\dim_{\Bbbk} H_q < \infty$ and admits a finite constant subdivision.

Example. $\Bbbk_{\mathbf{0}} \oplus \Bbbk[\mathbb{R}^2]$ admits constant regions $\{\mathbf{0}\}$ and $\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$

Example. Fix a poset *Q*.

- upset $U \subseteq Q$ if $U = \bigcup_{u \in U} Q_{\succeq u}$
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For any subset $S \subseteq Q$, set $\Bbbk[S] = \{\Bbbk_s\}_{s \in S}$.

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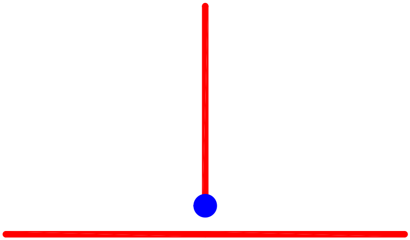

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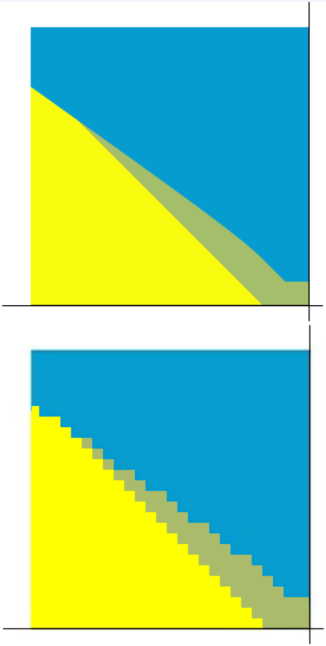

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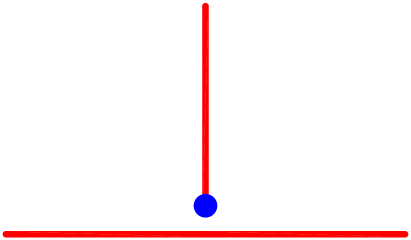

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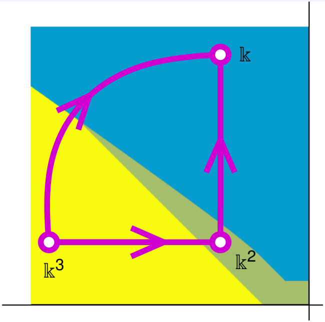


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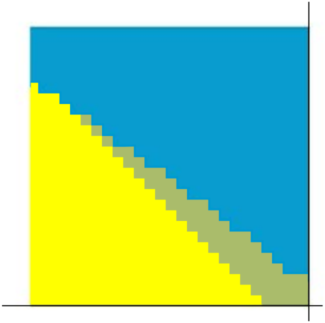

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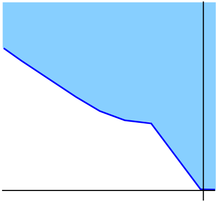
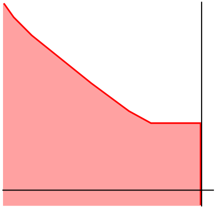
$U =$  and  $= D$

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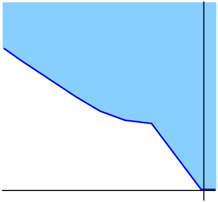
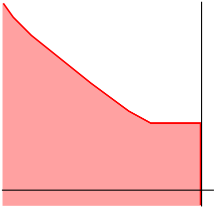
$U =$  and  $= D$

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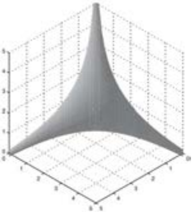
$U =$  or  $= D$

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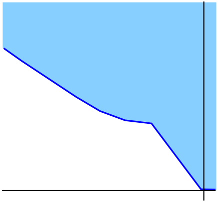
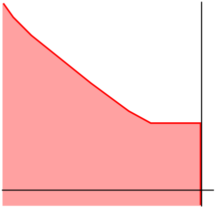
$U =$  and  $= D$

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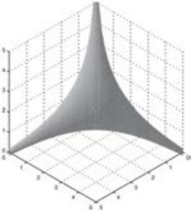
$U =$  or  $= D$

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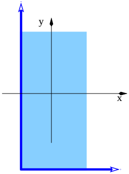


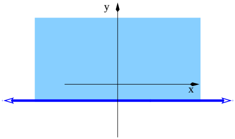


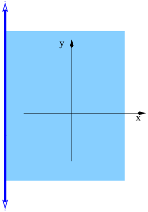


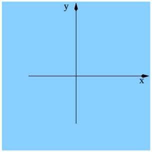

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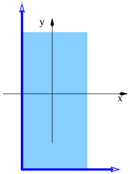

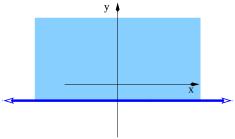

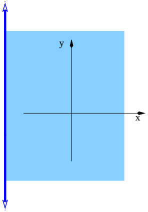

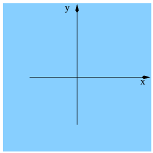


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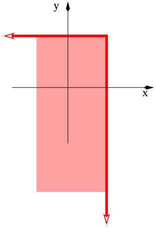

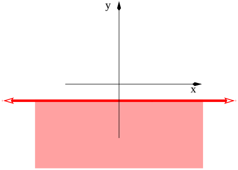

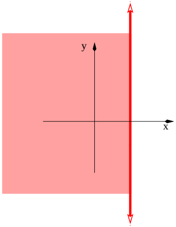

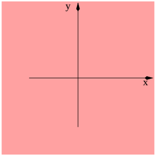

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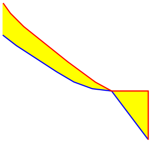

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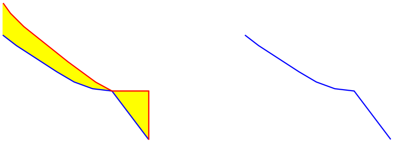

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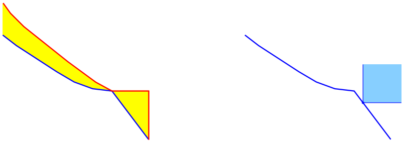

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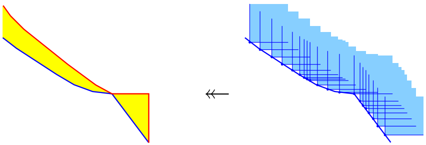

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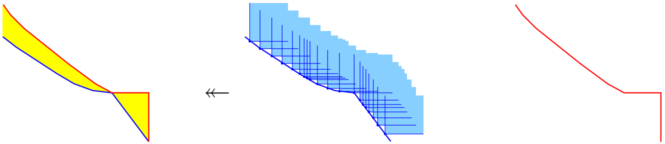

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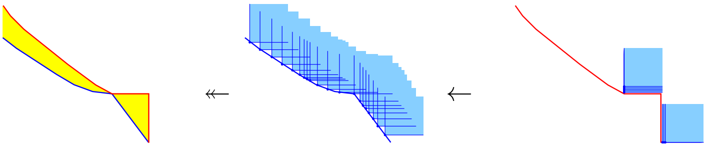

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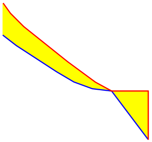
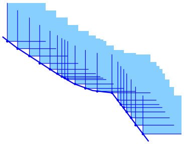


$\twoheadleftarrow$



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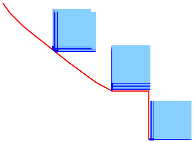

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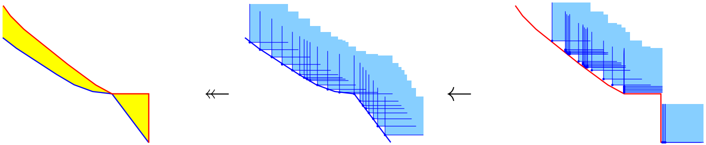

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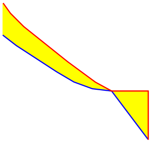

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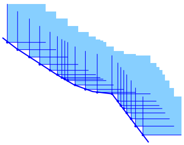

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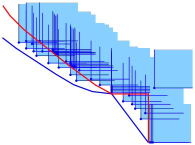

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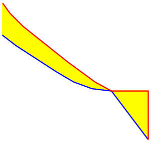
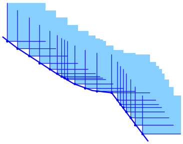
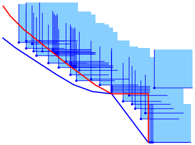
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

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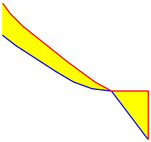
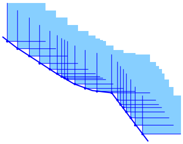
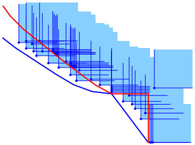
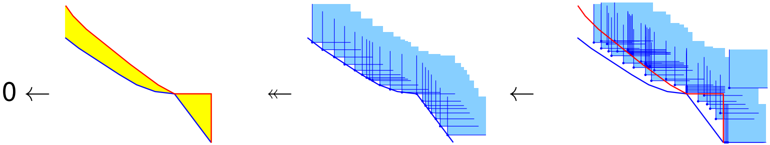
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

# Presenting poset modules

Over $\mathbb{R}^2$: free presentation



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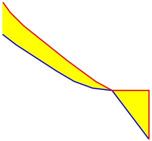

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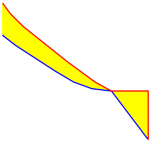
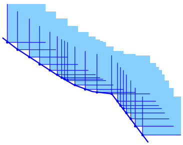
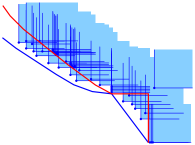
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

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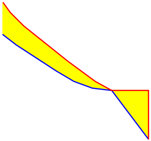

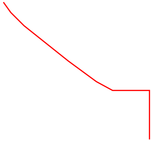

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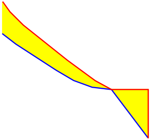
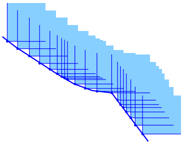


$\twoheadleftarrow$



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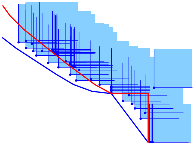

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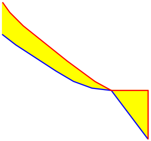

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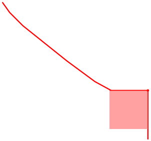

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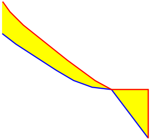
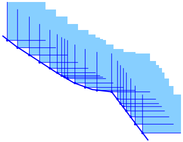
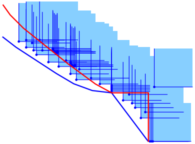
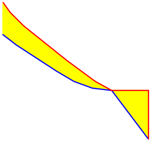
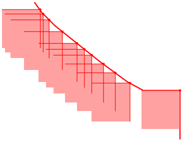
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

Over $\mathbb{R}^2$:

 $\hookrightarrow$ 

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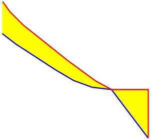
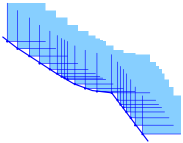
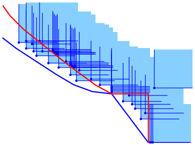
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

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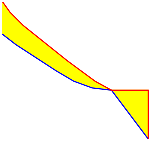
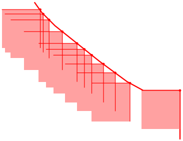
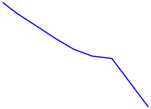
 $\hookrightarrow$  

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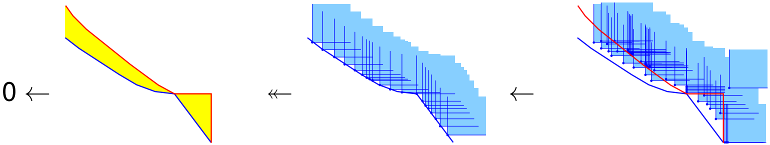
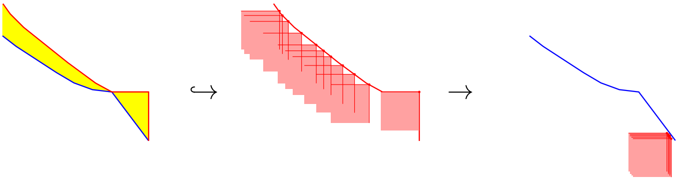



Over $\mathbb{R}^2$:



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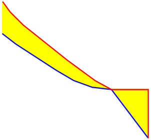
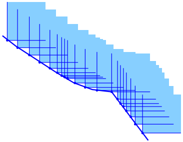
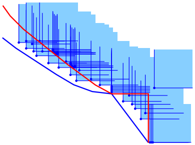
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

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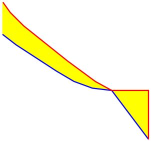
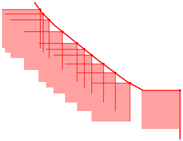
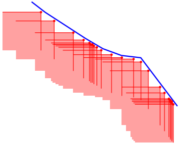
 $\hookrightarrow$  $\rightarrow$ 

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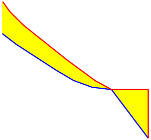
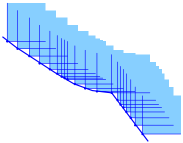
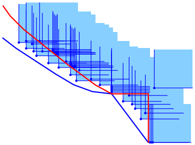
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

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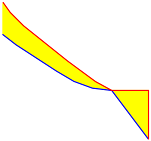
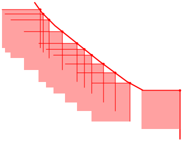
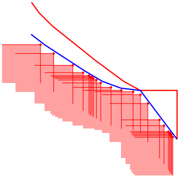
 $\hookrightarrow$  $\rightarrow$ 

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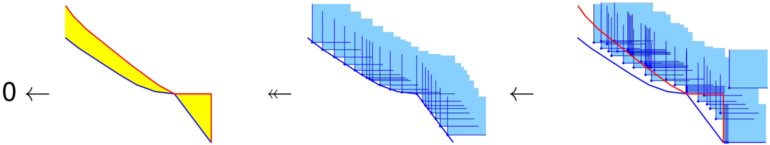


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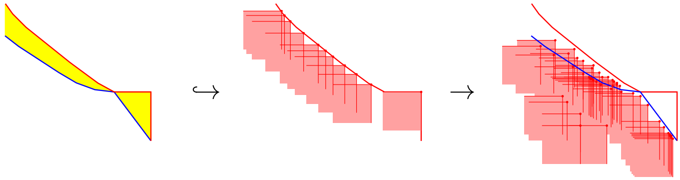

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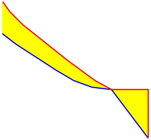
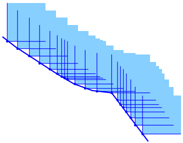
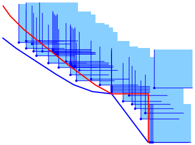
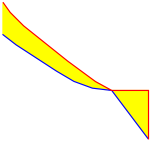
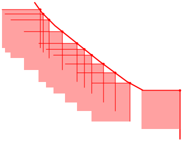
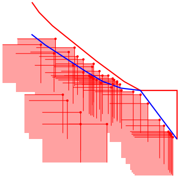
$0 \leftarrow$  $\twoheadleftarrow$  $\leftarrow$ 

Over $\mathbb{R}^2$: injective copresentation

$0 \rightarrow$  $\hookrightarrow$  $\rightarrow$ 

# Presenting poset modules

## Default.

- free presentation
- injective copresentation

**Def [–].** An $\mathbb{R}^n$-module $H$ is finitely encoded if it has either

- an upset presentation, namely a homomorphism

$$\Bbbk[U_1{}'] \oplus \cdots \oplus \Bbbk[U_\ell{}'] \to \Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k]$$

  with cokernel $\cong H$, or
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**Data structure:** monomial matrix

$$\Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k] \xrightarrow{\begin{array}{c|ccc} & D_1 & \cdots & D_\ell \\ \hline U_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ U_k & \varphi_{k1} & \cdots & \varphi_{k\ell} \end{array}} \Bbbk[D_1] \oplus \cdots \oplus \Bbbk[D_\ell]$$

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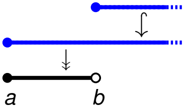
# Presenting poset modules

In $\mathbb{R}$:

has cokernel



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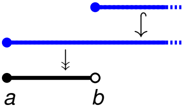


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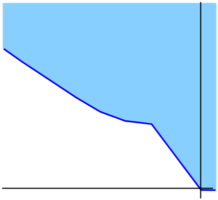
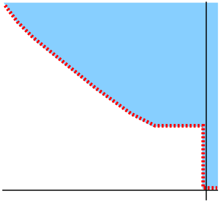


$\hookleftarrow$



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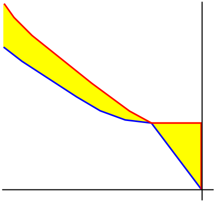

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$$\Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k] \longrightarrow \Bbbk[D_1] \oplus \cdots \oplus \Bbbk[D_\ell]$$

# Presenting poset modules

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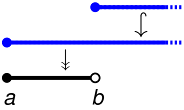
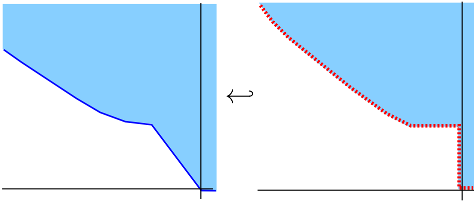
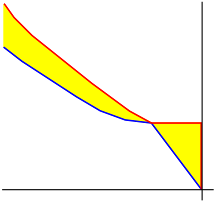



In $\mathbb{R}^2$:



has cokernel



# Presenting poset modules

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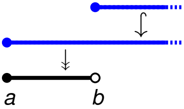
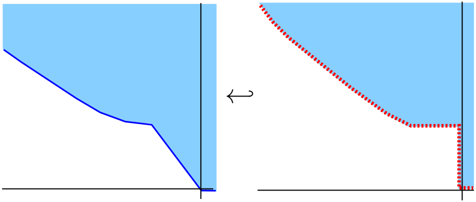
has cokernel



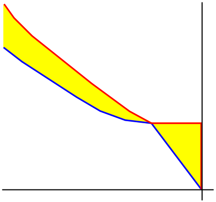

presentation: upset

In $\mathbb{R}^2$:



upset presentation

has cokernel



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In $\mathbb{R}$:

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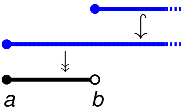
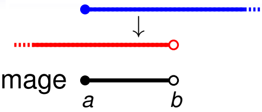
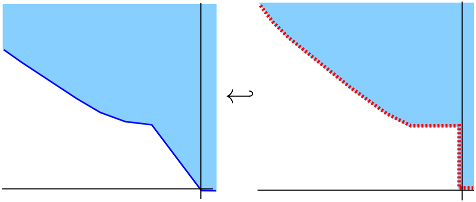



has image



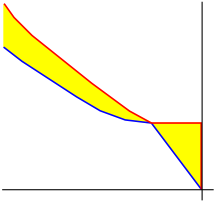

presentation: upset fringe

In $\mathbb{R}^2$:



upset presentation

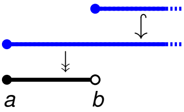
has cokernel



# Presenting poset modules

In $\mathbb{R}$:

has cokernel

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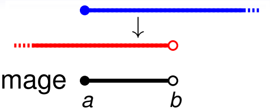
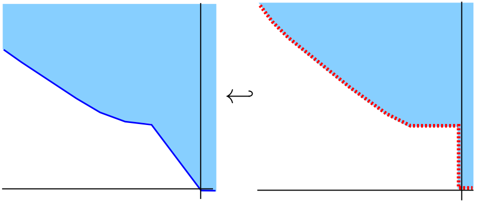


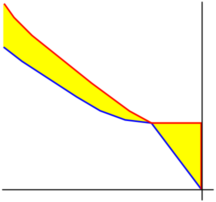

presentation: upset fringe

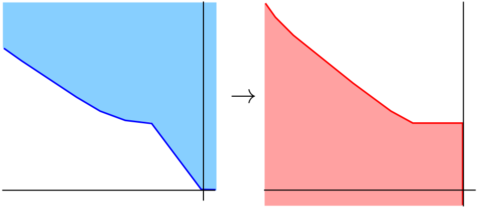
In $\mathbb{R}^2$:



upset presentation

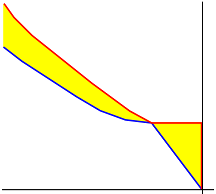
has cokernel





fringe presentation

has image



# Presenting poset modules

## Default.

- free presentation
- injective copresentation

**Def [–].** An $\mathbb{R}^n$-module $H$ is finitely encoded if it has either

- an upset presentation, namely a homomorphism

$$\Bbbk[U_1{}'] \oplus \cdots \oplus \Bbbk[U_\ell{}'] \to \Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k]$$

  with cokernel $\cong H$, or
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**Data structure:** monomial matrix

$$\Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k] \xrightarrow{\begin{array}{c|ccc} & D_1 & \cdots & D_\ell \\ \hline U_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ U_k & \varphi_{k1} & \cdots & \varphi_{k\ell} \end{array}} \Bbbk[D_1] \oplus \cdots \oplus \Bbbk[D_\ell]$$

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$$\begin{array}{cc} & \begin{array}{ccc} D_1 & \cdots & D_\ell \end{array} \quad \leftarrow \text{death downsets} \\ \text{birth upsets } \rightarrow \begin{array}{c} U_1 \\ \vdots \\ U_k \end{array} & \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & \ddots & \vdots \\ \varphi_{k1} & \cdots & \varphi_{k\ell} \end{bmatrix} \end{array}$$

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birth upsets $\rightarrow$ ; $\leftarrow$ death downsets ; $\leftarrow$ scalar entries

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# Syzygy theorem

**Thm [–].** A $Q$-finite module $H$ is finitely encoded if and only if it admits one, and hence all, of the following:

1. a finite constant subdivision (i.e., $H$ is tame); or
2. a finite fringe presentation; or
3. a finite upset presentation; or
4. a finite downset copresentation; or
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**Consequence.** (Homological) algebra of $H$ reduces to that of $\mathbb{k}[U]$ or $\mathbb{k}[D]$.

**Other commutative algebra.** Minimal primary decomposition

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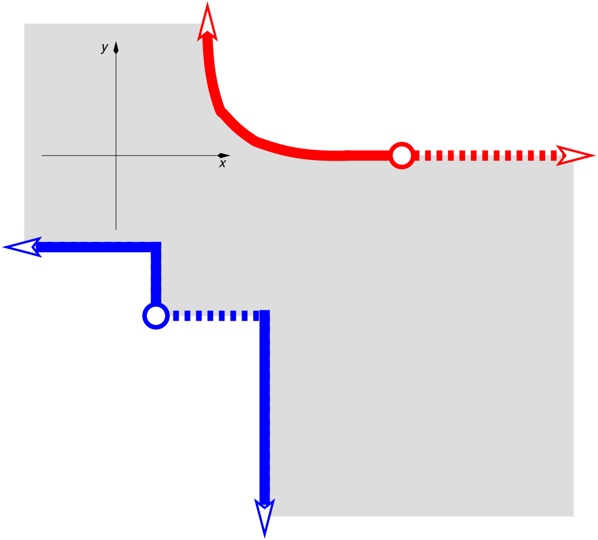

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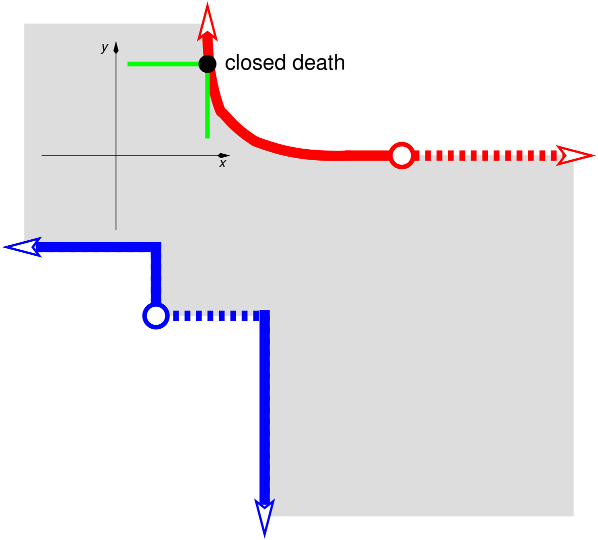

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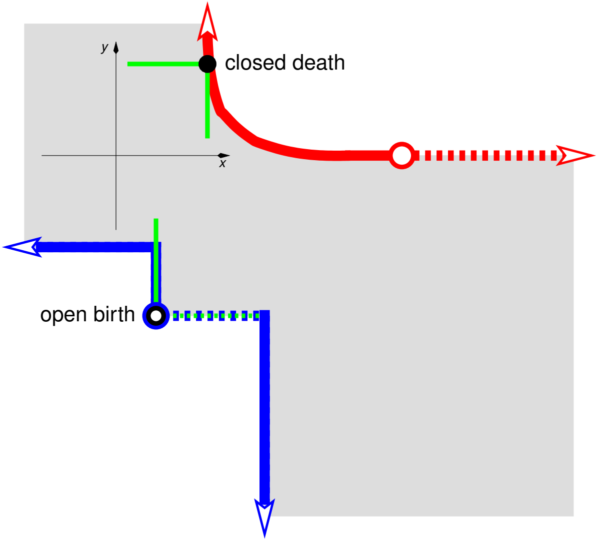

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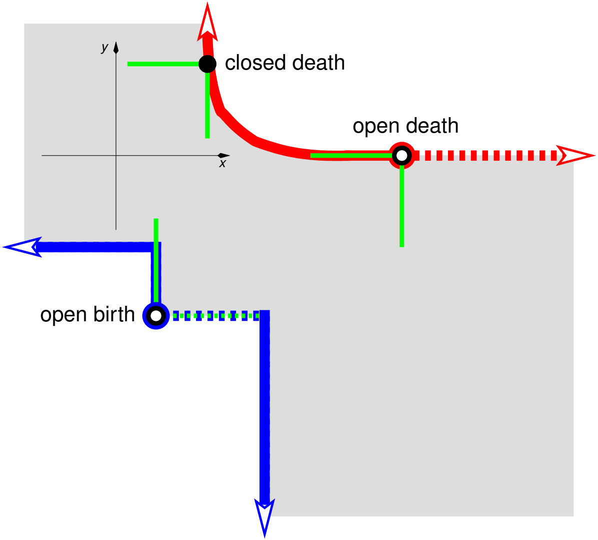

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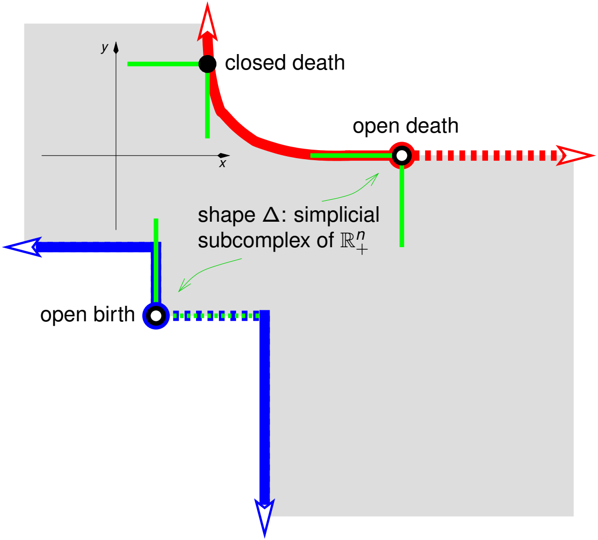

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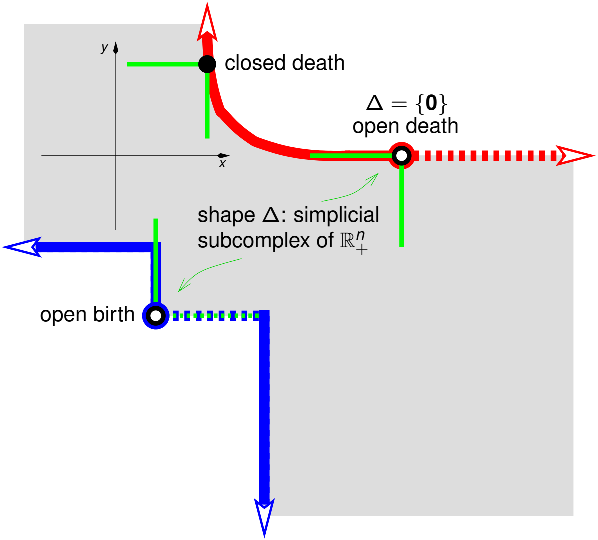

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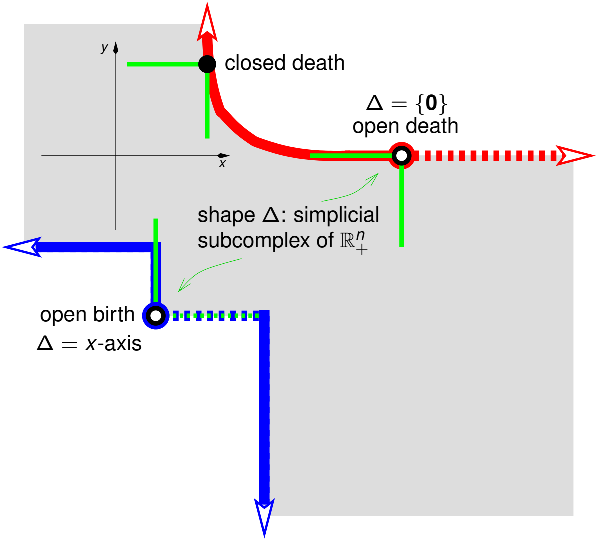

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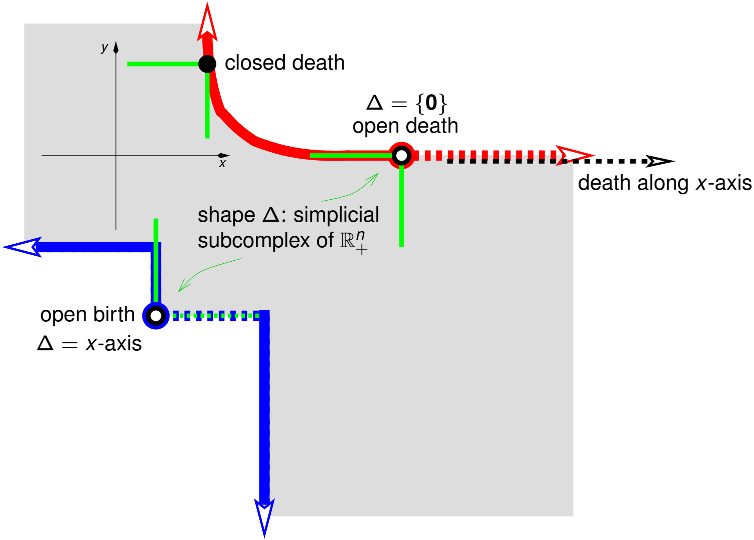

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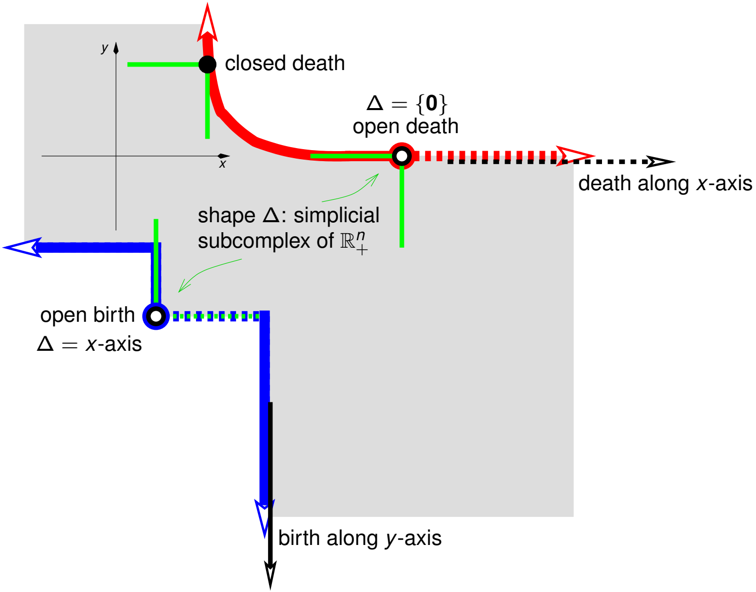

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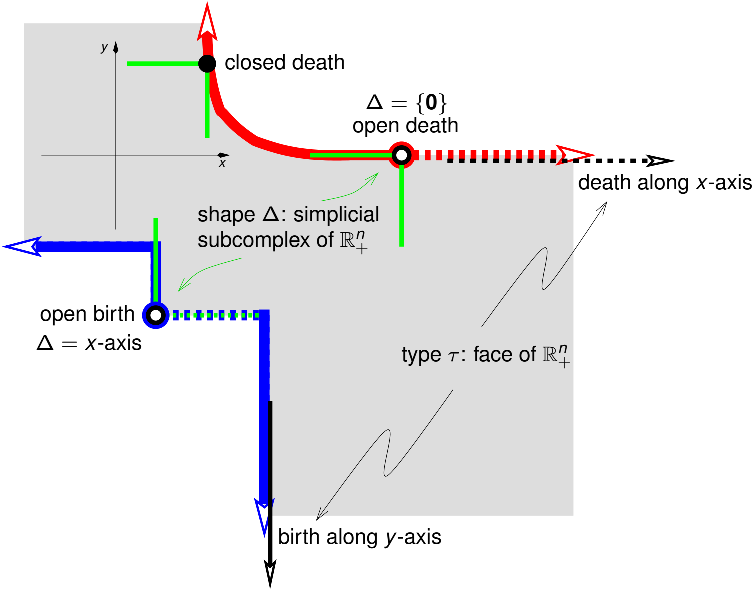
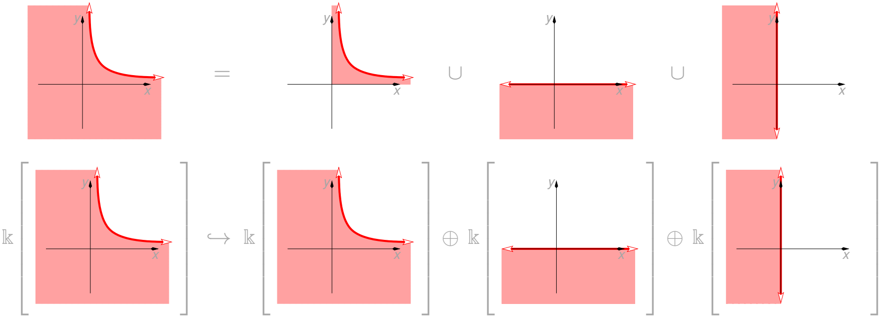


# Decomposition

**Def.** $\mathrm{soc}_\tau H$ = module of deaths along $\tau$ = upper boundary of $H$ parallel to $\tau$

**Def.** $H_\tau$ $\tau$-coprimary if all cogenerators have type $\tau$

**Thm [–].** Finitely encoded $\mathbb{R}^n$-modules admit minimal primary decomopsition: $H \hookrightarrow \bigoplus_{\text{faces } \tau} H_\tau$ with $H_\tau$ coprimary and $\mathrm{soc}_\tau H \xrightarrow{\sim} \mathrm{soc}_\tau H_\tau$

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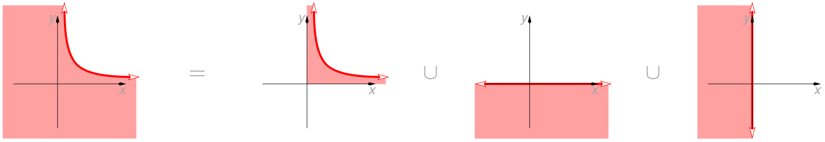


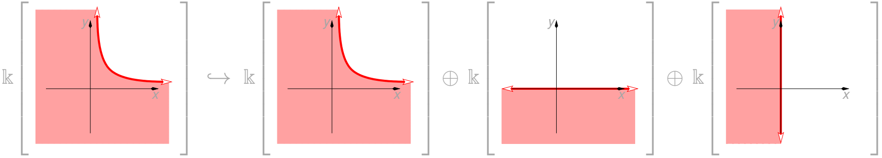


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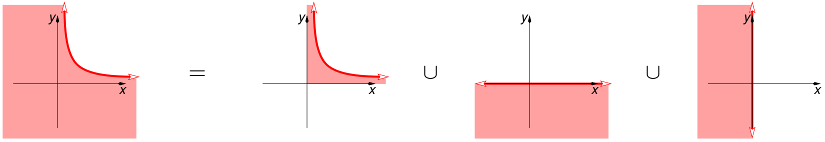


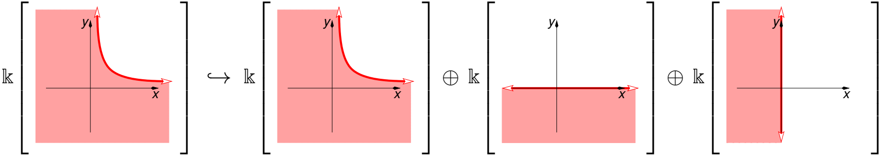


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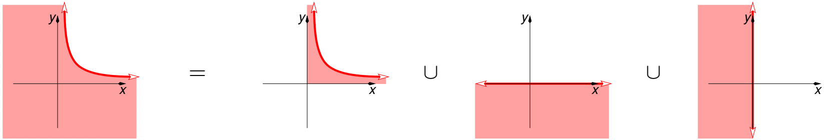


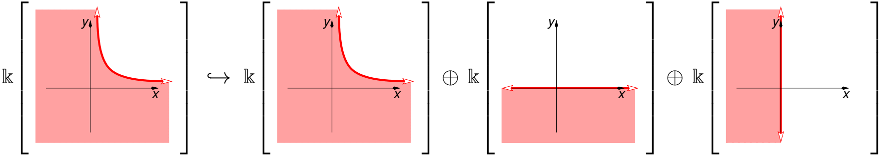


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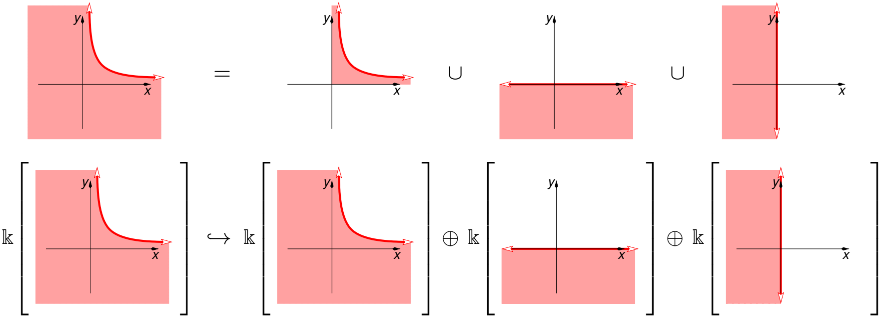


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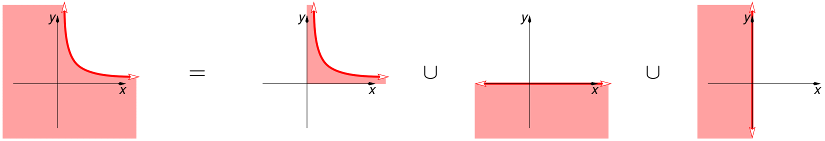


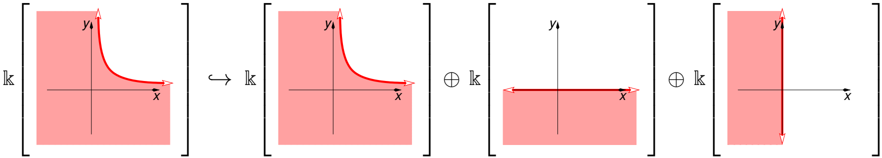


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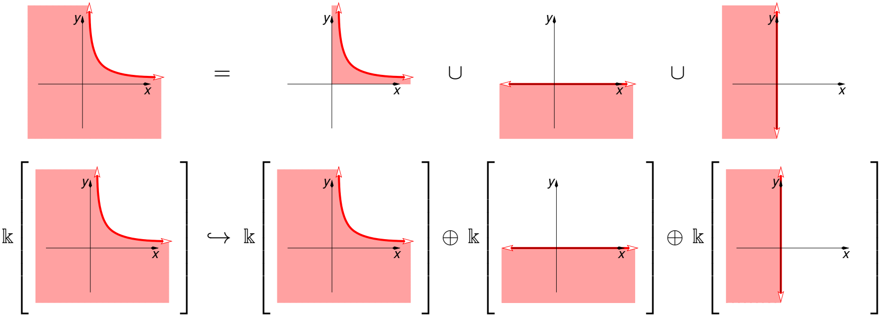


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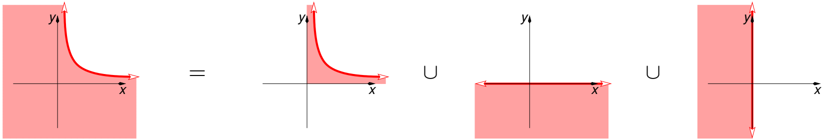


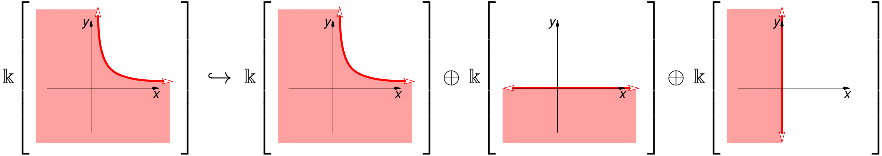


…but redundant! ( ← first crucial difference from finitely generated)
essential point:

**Thm [–].** $\varphi : M \hookrightarrow N \Leftrightarrow \operatorname{soc}_\tau \varphi : \operatorname{soc}_\tau M \hookrightarrow \operatorname{soc}_\tau N$ for all faces $\tau$.

**Pf.** Develop commutative algebra over $\Bbbk[\mathbb{R}^n_+]$ without noetherian hypotheses

# Next lectures

II. Tameness and encoding

- bar codes
- posets
- modules over posets
- constant subdivision
- tameness
- encoding

III. Presentation and decomposition: algebraic structures for computation

- covers and presentations: births
- hulls and copresentations: deaths
- primary decomposition
- fringe presentation: birth-and-death description
- equivalence of finiteness conditions

# Future directions

**Biology.** Test topological novelty hypothesis

- mark up wings with weird topology
- statistically analyze biparameter persistence summaries

**Commutative algebra**

- distance between modules via $K$-theory
- minimal resolution length in syzygy theorem

**Computation**

- calculate encoding / fringe presentation from vertices and Bézier curves
- homological algebra with (semialgebraic) upset and downset modules

**Topology.** Biparameter persistence as persistent intersection homology

**Algebraic geometry.** Kashiwara–Schapira conjecture: finite encoding as $\lambda$-stratification of constructible sheaf in $\gamma$-topology

**Probability.** Bootstrapping persistent homology: compare homology from one (large) dataset to homology from (relatively) few small subsamples

- feature strength $\leftrightarrow$ likelihood of appearing in appropriate-size subsample
- parallelize (persistent) homology computations
- yield probabilistic answers to probabilistic questions

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Reboot at 14:00