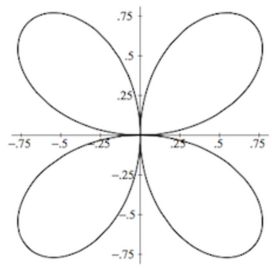
Varieties

5ta ESCUELA DE ANALISIS TOPOLOGICO DE DATOS, November 22
Reference: Cox, Little, O'Shea - Ideals, varieties and algorithms

1. One of the prettiest examples from polar coordinates is the four-leaved rose



This curve is defined by the polar equation $r = \sin(2\theta)$. We will show that this curve is an affine variety.

   (a) Using $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$ show that the four-leaved rose is contained in the affine variety $\mathbf{V}((x^2 + y^2)^3 - 4x^2y^2)$.

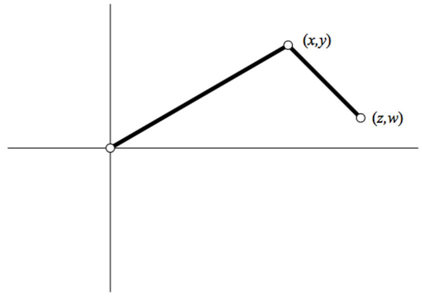
   (b) Now argue that $\mathbf{V}((x^2 + y^2)^3 - 4x^2y^2)$ is contained in the four-leaved rose.

   Combining (a) and (b) we have proved that the four-leaved rose is the affine variety $\mathbf{V}((x^2 + y^2)^3 - 4x^2y^2)$.

2. It can take some work to show that something is not an affine variety. For example, consider the set
$$X = \{(x, x) : x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2,$$
which is the straight line $x = y$ with the point $(1, 1)$ removed. To show that X is not an affine variety, suppose that $X = \mathbf{V}(f_1, \ldots, f_s)$.Then each f_i vanishes on X, and if we can show that f_i also vanishes at $(1, 1)$, we will get the desired contradiction. Thus, here is what you are to prove : if $f \in \mathbb{R}[x, y]$ vanishes on X, then $f(1, 1) = 0$.

   Hint: Let $g(t) = f(t, t)$, which is a polynomial in $\mathbb{R}[t]$.

3. To give an idea of some of the applications of affine varieties, let us consider a simple example from robotics. Suppose we have a robot arm in the plane consisting of two linked rods of lengths 1 and 2, with the longer rod anchored at the origin:



The "state" of the arm is completely described by the coordinates (x, y) and (z, w) indicated in the figure. Thus the state can be regarded as a 4-tuple $(x, y, z, w) \in \mathbb{R}^4$. However, not all 4-tuples can occur as states of the arm. In fact, it is easy to see that the subset of possible states is the affine variety in 4 defined by the equations

$$\begin{aligned} x^2 + y^2 &= 4, \\ (x - z)^2 + (y - w)^2 &= 1. \end{aligned}$$

Now consider a robot arm in 2 that consists of three arms of lengths 3, 2, and 1, respectively. The arm of length 3 is anchored at the origin, the arm of length 2 is attached to the free end of the arm of length 3, and the arm of length 1 is attached to the free end of the arm of length 2. The "hand" of the robot arm is attached to the end of the arm of length 1.

(a) Draw a picture of the robot arm.

(b) How many variables does it take to determine the "state" of the robot arm?

(c) Give the equations for the variety of possible states.

(d) If (u, v) is the position of the hand, explain why $u^2 + v^2 \leq 36$.

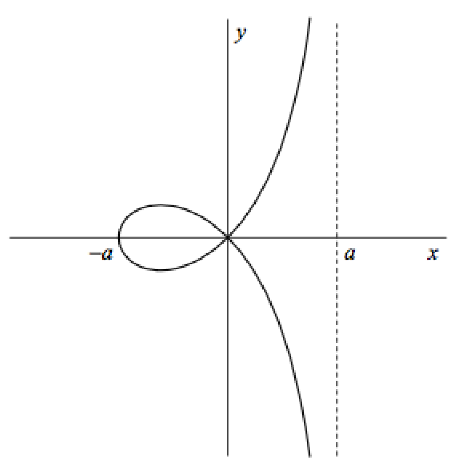
(e) Suppose we "lock" the joint between the length 3 and length 2 arms to form a straight angle, but allow the other joint to move freely. Draw a picture to show that in these configurations, (u, v) can be any point of the annulus $16 \leq u^2 + v^2 \leq 36$.

(f) Draw a picture to show that (u, v) can be any point in the disk $u^2 + v^2 \leq 36$. Hint: These positions can be reached by putting the second joint in a fixed, special position.

4. The **strophoid** is a curve that was studied by various mathematicians, including Isaac Barrow (1630—1677), Jean Bernoulli (1667—1748), and Maria Agnesi (1718—1799). A trigonometric parametrization is given by

$$\begin{aligned} x &= a\sin(t), \\ y &= a\tan(t)(1 + \sin(t)) \end{aligned}$$

where a is a constant. If we let t vary in the range $-4.5 \leq t \leq 1.5$, we get the picture shown below:



Find the equation in x and y that describes the strophoid. Hint: If you are sloppy, you will get the equation $(a^2 - x^2)y^2 = x^2(a + x)^2$. To see why this is not quite correct, see what happens when $x = -a$.